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Comparing Supply Function Equilibria of Pay-as-Bid and Uniform-Price Auctions

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Abstract

This paper derives a Supply Function Equilibrium (SFE) of a pay-as-bid auction, also called discriminatory auction. Such an auction is used in the balancing market for electric power in Britain. For some probability distributions of demand a pure-strategy equilibrium does not exist. If demand follows an inverse polynomial probability distribution, SFE always exists. Assuming this probability distribution, the pay-as-bid procurement auction is compared to a SFE of a uniform-price procurement auction, the auction form of most electric power markets. The demand-weighted average price is found to be equal or lower in the pay-as-bid procurement auction.

Keywords: supply function equilibrium, pay-as-bid auction, uniform-price auction, discriminatory auction, oligopoly, capacity constraint, wholesale electricity market

JEL codes: C62, D43, D44, L11, L13, L94

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1. INTRODUCTION

Before 2001, essentially all electric power markets have been organized as uniform-price auctions (UPA). In 2001, electricity trading in the balancing market of England and Wales switched from a UPA to a pay-as-bid auction (PABA), also called discriminatory auction. It was the belief of the British regulatory authority (Ofgem) that the reform would decrease mark-ups in wholesale electricity prices. Before the collapse of the California Power Exchange, a similar switch was considered also for this market [16].

The balancing market allows the system operator to buy or sell last-minute power to keep a continuous balance of demand and supply. This paper focuses on market situations where more supply is needed, i.e. the system operator buys power as in a procurement auction. However, it is straightforward to show analogous results for market situations where less supply is needed and the system operator sells power in the balancing market, i.e. it is a sales auction.

In a UPA, all accepted bids are paid the marginal-bid. Thus in its procurement version, all infra-marginal bids are accepted at a price above their bid. In a PABA, all accepted bids are paid their bid. A naive first assumption is that pay-as-bid auctions would drastically reduce mark-ups for infra-marginal units and thereby decrease average electricity prices. However, firms will change their bidding strategy after a switch to PABA. Using intuition and experience from classical auction theory, many papers actually argue in favour of UPA, see e.g. Kahn et al. [16] and Wolfram [23]. There is also an experiment by Rassenti et al. [20], which suggests that average prices are higher in PABA[3].

In classical auction theory, i.e. the *private value*, *common value* and *affiliated value* models, the demand by the auctioneer is certain, whereas there are uncertainties in costs [19]. In electricity markets, however, production costs are often well-known, but the imbalance—the demand by the system operator—is uncertain, as it depends on unexpected temperature variations and unexpected outages in generators, machines and transmission-lines. Thus costs are often assumed to be certain and the demand uncertain in models of strategic bidding in electricity procurement auctions. Most theoretical studies of electric power auctions have been devoted to the UPA, see e.g. [1-3,9,11,13-15,21]. There are, however, two recent papers

[3] The demand in the experiment is not told to the players, but is certain in each period and the players can figure it out while playing. As in SFE with certain demand, this set-up would lead to an enormous range of equilibria [17]. Therefore the experimental results are not directly comparable to the results in this paper nor to previous theoretical comparisons of UPA and PABA in the electricity market.

that have studied bidding behaviour in electric power markets organized as PABA and compared prices and welfare in PABA and UPA [6,8]. Both of them give some support for PABA.

Federico and Rahman [8] compare UPA and PABA for two polar cases, perfect competition and monopoly, assuming that demand is elastic and follows a uniform probability distribution. They show that expected output decreases and expected consumer surplus increases after a switch to a PABA. On the other hand, welfare is reduced in the competitive case. Under monopoly bidding, welfare is larger in PABA, if and only if marginal costs are sufficiently flat and demand uncertainty sufficiently low.

Fabra et al. [6] derive a Nash equilibrium for a duopoly model with constant marginal costs. Further, each producer must submit a horizontal (perfectly elastic) bid for its entire capacity. Firms are asymmetric in terms of both marginal costs and capacity. Further, demand is inelastic and known with certainty by the producers. Under these circumstances they show that average prices are lower in the PABA than in a UPA. Numerical examples suggest that the difference might be substantial. The implications for production efficiency are ambiguous, and depend on parameter values. Further, if demand is sufficiently high, the PABA has no pure strategy equilibria, only a mixed strategy equilibrium. They make extensions of the model, but they do not lead to any definite conclusions with regard to the comparison of PABA and UPA.

The Supply Function Equilibrium (SFE) under uncertainty was introduced by Klemperer & Meyer [17]. The set-up of their model is similar to the organization of electricity markets, as firms submit supply functions to a uniform-price auction with uncertain demand. In the non-cooperative Nash equilibrium of the static game, each producer commits to the bid that maximizes his expected profit given the bids of his competitors and the properties of the uncertain demand. SFE is an often used model of strategic bidding in electric power markets organized as UPA [1-3,11,13-15,21]. In this paper, the fundamental assumptions of the SFE model are used to derive a similar model for a pay-as-bid auction. It is assumed that demand is inelastic and that there is risk of power shortage, which could be arbitrarily small. Both assumptions are realistic for balancing markets [13]. As in [13], the risk of power shortage ensures a unique equilibrium. To facilitate an analytical solution, only symmetric equilibria are considered. As for UPA, it should be possible to extend the analysis to consider asymmetric producers [14,15].

Another contribution of this paper is the comparison of the two SFE models for procurement auctions. In case of the inverse probability distribution, the demand-weighted average price is weakly lower in PABA compared to UPA[4]. In a one-shot game with inelastic demand and symmetric firms, mark-ups have no implications for social efficiency. Still, large mark-ups imply a large redistribution of income, from power consumers to power producers, which would be of social interest. Further, high mark-ups would, in the long term, lead to welfare losses due to unnecessary investment in additional capacity by firms entering the market [11].

The notation and assumptions are presented in Section 2. The unique SFE of a PABA is derived in Section 3. A first-order condition is calculated, which implies that the bid of each production unit is chosen to maximize its expected profit, given the bids of the competitors. The risk of power shortage gives an end-condition for the supply functions. There is a unique equilibrium candidate that satisfies both the first-order condition and the end-condition. Next, a second-order condition is derived. It is observed that for some combinations of marginal costs and probability distributions of demand, there are no pure strategy equilibria. On the other hand, there is always a unique equilibrium for the inverse polynomial probability distribution, a generalized Zipfian distribution [24]. In Section 4, average prices in the two procurement auctions are compared, assuming that demand follows this probability distribution. In Section 5, the two supply function equilibria are illustrated with a simple example. The paper is concluded in Section 6.

2. NOTATION AND ASSUMPTIONS

Assume that there are $N \geq 2$ symmetric producers. The bid of each producer i consists of a monotonically increasing supply function $S_i(p)$, where p is the price[5]. The inverse of the supply function is denoted by $p_i(S_i)$. $S_{-i}(p)$ and $S(p)$ are the competitors' total supply and the total supply, respectively. As in the original work by Klemperer & Meyer [17], only equilibria with twice continuously differentiable supply functions are considered. Thus in equilibrium, $p_i(S_i)$ is smooth for $S_i \in \left(0, \bar{\varepsilon}/N\right)$, where $\bar{\varepsilon}$ is the total capacity of all producers.

Denote the inelastic demand by ε, its probability density function by $f(\varepsilon)$ and its distribution function by $F(\varepsilon)$. The density function is continuously differentiable and has support on the interval $\left[0, \hat{\varepsilon}\right]$, where $\hat{\varepsilon}$ is maximum demand. It is assumed that $\hat{\varepsilon} \geq \bar{\varepsilon}$, i.e. the

[4] Analogously, demand-weighted average prices in sales auctions would be higher in PABA compared to UPA.
[5] Electricity auctions do normally not accept decreasing supply functions. Further, in most cases it would not be in a firm's interest to submit decreasing supply functions, as it would not be consistent with profit maximization [17].

capacity constraints of all producers will bind with a positive probability, which could be arbitrarily small. Demand is zero above the reservation price (price cap) $\overline{p}$. Thus the market price equals the price cap, in case of extreme outcomes where demand exceeds the market capacity.

All firms have identical cost functions $C(S_i)$, which are increasing, convex, twice continuously differentiable, and fulfill $C'(\overline{\varepsilon} / N) < \overline{p}$.

The price of the marginal unit as a function of demand is denoted $p(\varepsilon)$. The average price as a function of demand, $\hat{p}(\varepsilon)$, is also called the equilibrium price. In UPA all accepted bids are paid the marginal bid, i.e. $p_U(\varepsilon) = \hat{p}_U(\varepsilon)$. R denotes the sum of firms' expected revenues and π_i denotes the expected profit of firm i.

3. THE UNIQUE SYMMETRIC SFE OF A PAY-AS-BID AUCTION

In the SFE of a UPA, a firm chooses — given its residual demand — a supply function, such that his profit is maximized for each demand outcome [17]. In Section 3.1, a first-order condition of PABA is derived. It implies that a firm chooses a supply function, such that the expected profit is maximized for each of his production units, given the residual demand. The first-order condition is a differential equation, which can be solved for general cost functions. The solution has one arbitrary constant. In Section 3.2, this constant is pinned down by considering the risk of power shortage. The symmetric equilibrium bids of all firms must reach the price cap exactly when the capacity constraint binds. This is the end-condition.

The first-order condition and the end-condition must necessarily be satisfied in equilibrium. In Section 3.3, a sufficient second-order condition is derived. Unlike UPA, it is not possible to prove that all increasing smooth supply functions satisfying the necessary conditions are SFE of PABA. In particular, it turns out that if there is some demand, for which $C'(\varepsilon / N)$ is constant and $f'(\varepsilon) \geq 0$, then pure strategy equilibria of PABA do not exist. Further, I show that a pure strategy equilibrium always exists if demand follows an inverse polynomial probability distribution, for which the inverse of the hazard rate is linear. For this particular distribution, the PABA first-order condition is similar to the UPA first-order condition.

3.1. The first-order condition

It is assumed that the competitors of firm i follow a symmetric equilibrium candidate. The first-order condition, which is derived below, must necessarily be fulfilled, if the strategy implied by the symmetric equilibrium candidate locally maximizes firm i's expected profit. To avoid differentiability problems all considered deviations of firm i satisfies $p_i(0) = p_j(0)$ and $p_i(\bar{\varepsilon}/N) = p_j(\bar{\varepsilon}/N)\ \forall j \neq i.$ The profit from an accepted bid of an infinitesimally small unit is $\left[p(S_i) - C'(S_i)\right]dS_i.$ Thus the expected profit of firm i is:

$$\pi_i = \int_0^{\bar{\varepsilon}} f(\varepsilon) \int_0^{\varepsilon - S_{-i}(p(\varepsilon))} \left[p_i(S_i) - C'(S_i)\right] dS_i d\varepsilon + \int_{\underline{\varepsilon}}^{\hat{\varepsilon}} f(\varepsilon) \int_0^{\bar{\varepsilon}/N} \left[p_i(S_i) - C'(S_i)\right] dS_i d\varepsilon.$$

The second term is the contribution from demand outcomes exceeding market capacity. By changing the order of integration [22], the following can be shown:

$$\begin{aligned} \pi_i[p_i(S_i)] &= \int_0^{\bar{\varepsilon}/N} \left[p_i(S_i) - C'(S_i)\right] \int_{S_i + S_{-i}[p_i(S_i)]}^{\bar{\varepsilon}} f(\varepsilon) d\varepsilon dS_i + \\ &+ \int_0^{\bar{\varepsilon}/N} \left[p_i(S_i) - C'(S_i)\right] \int_{\underline{\varepsilon}}^{\hat{\varepsilon}} f(\varepsilon) d\varepsilon dS_i = \int_0^{\bar{\varepsilon}/N} \left[p_i(S_i) - C'(S_i)\right] \int_{S_i + S_{-i}[p_i(S_i)]}^{\hat{\varepsilon}} f(\varepsilon) d\varepsilon dS_i = \\ &= \int_0^{\bar{\varepsilon}/N} \underbrace{\left[p_i(S_i) - C'(S_i)\right]\left[1 - F\left(S_i + S_{-i}[p_i(S_i)]\right)\right]}_{\varphi_i[p_i(S_i), S_i]} dS_i. \end{aligned} \tag{1}$$

Firm i chooses his bid function $p_i(S_i)$ such that his expected profit is maximized, i.e. the firm is facing a degenerate calculus of variation problem with the fixed terminal points $p_i(0) = p_j(0)$ and $p_i(\bar{\varepsilon}/N) = p_j(\bar{\varepsilon}/N)$. As there is no $p_i'(S_i)$ in the integral, the Euler equation is simply [5]:

$$\begin{aligned} &\frac{\partial \varphi_i}{\partial p_i} = 1 - F\left[S_{-i}(p_i(S_i)) + S_i\right] - S_{-i}'(p_i(S_i))(p_i(S_i) - C'(S_i)) f\left[S_{-i}(p_i(S_i)) + S_i\right] = 0, \\ &\forall\, S_i \in \left[0, \bar{\varepsilon}/N\right] \end{aligned} \tag{2}$$

The functional $\varphi_i[p_i(S_i), S_i]$ equals the contribution to the expected profit from an infinitesimally small unit. Thus the Euler equation implies that the expected profit from each unit is maximized, given the residual demand. Only equilibria with smooth and increasing supply functions are considered. Thus (2) can be written:

$$1-F[S_{-i}(p)+S_i(p)]-S_{-i}'(p)(p-C'(S_i(p)))f[S_{-i}(p)+S_i(p)]=0,$$
$$\forall\, p: S_i(p)\in\left(0,\bar{\varepsilon}/N\right) \qquad (3)$$

Only symmetric SFE are considered, i.e. $S_{-i}(p)\equiv(N-1)S_i(p)$. Thus

$$1-F[NS_i(p)]-(N-1)S_i'(p)(p-C'(S_i(p)))f[NS_i(p)]=0, \forall\, p: S_i(p)\in\left(0,\bar{\varepsilon}/N\right) \qquad (4)$$

This first-order condition is henceforth referred to as PABA FOC. It corresponds to the first-order condition of UPA, which has been derived by Klemperer & Meyer [17]. Note that (4) implies $S_i'(p)>0$ for $p>C'$.

In order to solve the differential equation, it is transformed into a differential equation with $p(\varepsilon)$, the price of the marginal unit as a function of the demand, instead of $S_i(p)$. The same trick is used when solving the differential equation associated with the SFE for UPA [1,21]. In the symmetric equilibrium, $\varepsilon = NS_i(p(\varepsilon))$ and $S_i' = \frac{1}{Np'(\varepsilon)}$, if $\varepsilon \le \bar{\varepsilon}$. Thus:

$$1-F(\varepsilon)-\frac{(N-1)}{Np'(\varepsilon)}[p(\varepsilon)-C'(\varepsilon/N)]f(\varepsilon)=0 \quad \forall \varepsilon\in\left[0,\bar{\varepsilon}\right]$$

and

$$\frac{(N-1)p(\varepsilon)f(\varepsilon)}{N}-p'(\varepsilon)[1-F(\varepsilon)]=\frac{(N-1)C'(\varepsilon/N)f(\varepsilon)}{N}.$$

This differential equation can be solved by means of the integrating factor $[1-F(\varepsilon)]^{\frac{N-1}{N}-1}$.

$$p(\varepsilon)=\frac{A-\int^{\varepsilon}(N-1)C'(u/N)f(u)[1-F(u)]^{\frac{N-1}{N}-1}du}{N[1-F(\varepsilon)]^{\frac{N-1}{N}}}, \qquad (5)$$

where A is an arbitrary constant.

3.2. Determining the arbitrary constant

Analogous to SFE of UPA, the arbitrary constant A allows for a continuum of potential equilibria [17]. In this section, I argue that the arbitrary constant can be uniquely determined if, as in [13], the capacity constraint binds with a positive probability, i.e. there is a risk of large demand shocks and/or unexpected multiple generation failures[6]. Then the price of the marginal unit must reach the price cap exactly when the capacity constraint starts to bind.

As noted in Section 2, the analysis is confined to equilibria with twice continuously differentiable supply functions. Hence, $S_i'(p) < \infty,$ which implies that $p'(\varepsilon) > 0$ [7]. Thus by construction, if the price of the marginal unit reaches the price cap before the total capacity binds, i.e. $p(\varepsilon^*) = \bar{p}$ for $\varepsilon^* < \bar{\varepsilon},$ some capacity is withheld from the auction. It cannot be optimal to withhold power. A producer will find it profitable to slightly reduce his bids, such that the previously withheld units are offered just below the price cap. Bidding with his whole capacity will significantly increase the contribution to the expected profit from demand outcomes $\varepsilon > \varepsilon^*$, while the possible profit reductions for demand outcomes $\varepsilon \le \varepsilon^*$ can be made arbitrarily small.

The highest bid in the auction must equal the price cap. Otherwise, the highest bid can be significantly increased without lowering the probability that its associated unit is accepted. In summary, the price of the marginal unit must reach the price cap, but not before the capacity constraint binds. Hence, the arbitrary constant can be pinned down by the end-condition $p(\bar{\varepsilon}) = \bar{p}$ [8]. Thus it follows from (5) that:

$$p(\varepsilon) = \frac{N\left[1 - F(\bar{\varepsilon})\right]^{\frac{N-1}{N}}\bar{p} + \int_{\varepsilon}^{\bar{\varepsilon}} (N-1)C'(u/N)f(u)\left[1 - F(u)\right]^{\frac{N-1}{N}-1} du}{N\left[1 - F(\varepsilon)\right]^{\frac{N-1}{N}}}, \text{ if} \varepsilon \ge 0. \qquad (6)$$

3.3. The second-order condition

The only remaining equilibrium candidate is given by (6) and fulfills both PABA FOC and the end-condition. For this candidate, let $\breve{p}(\varepsilon)$ be the price of the marginal unit as a function of demand. The symmetric supply functions of the candidate are designated by $\breve{S}_i(p)$. If all firms have supply functions equal to $\breve{S}_i$, then—due to PABA FOC—the expected profit of

[6] To avoid inconsistencies in the model, production uncertainties are only considered for firms that do not bid in the balancing market. Two examples of such firms in the British market are British Energy and British nuclear group, who both have nuclear power exclusively.

[7] The assumption simplifies the proof, but is not critical. Allowing for horizontal (perfectly elastic) bids would not change the result. As in a Bertrand game, it is profitable to slightly undercut competitors' horizontal bids [13].

[8] The same end-condition is used to derive a unique SFE for UPA [13]. Baldick & Hogan have suggested the same end-condition for UPA, but with a weaker motivation. The price cap and capacity constraints can be seen as public signals that will coordinate the bids of the producers [2].

any unit of any firm is at a local extremum. By studying the second-order condition, it can be verified that, under certain conditions, the expected profits are globally maximized for all production units. This is a sufficient condition for a SFE.

For a given S_i, the expected profit from the marginal unit of firm i is, see (1):

$$\varphi_i(S_i, p) = [p - C'(S_i)][1 - F(S_i + S_{-i}(p))], \tag{7}$$

where $p=p_i(S_i)$ is the bid of the marginal unit. As the competitors follow $\breve{S}_i$, (2) can be written:

$$\frac{\partial \varphi_i(S_i, p)}{\partial p} = 1 - F\left[\breve{S}_{-i}(p) + S_i\right] - \breve{S}_{-i}{}'(p)(p - C'(S_i)) f\left[\breve{S}_{-i}(p) + S_i\right]$$

or

$$\frac{\partial \varphi_i(S_i, p)}{\partial p} = f\left[\breve{S}_{-i}(p) + S_i\right] \underbrace{\left\{ \frac{\overbrace{1 - F\left[\breve{S}_{-i}(p) + S_i\right]}^{G\left[\breve{S}_{-i}(p)+S_i\right]}}{f\left[\breve{S}_{-i}(p) + S_i\right]} - \breve{S}_{-i}{}'(p)(p - C'(S_i)) \right\}}_{H(p,S_i)}, \tag{8}$$

where $G(x)$ is the inverse of the hazard rate. Let $p^* = \breve{p}_i(S_i)$. PABA FOC guarantees that $\left.\frac{\partial \varphi_i(S_i, p)}{\partial p}\right|_{p=p^*} = 0$. Now if $\frac{\partial \varphi_i(S_i, p)}{\partial p} > 0$ for $p \in \left[\breve{p}(0), p^*\right)$[9] and $\frac{\partial \varphi_i(S_i, p)}{\partial p} < 0$ for $p \in \left(p^*, \overline{p}\right]$, then $\varphi_i(S_i, p)$ is globally maximized at the price p^*. As $f>0$ and $H\left(p^*, S_i\right) = 0$, it is sufficient to show that $\frac{\partial H(p, S_i)}{\partial p} < 0$ for $p \in \left[\breve{p}(0), \overline{p}\right]$

$$\frac{\partial H}{\partial p} = G'\left[\breve{S}_{-i}(p) + S_i\right]\breve{S}'_{-i}(p) - \breve{S}_{-i}{}''(p)(p - C'(S_i)) - \breve{S}_{-i}{}'(p). \tag{9}$$

An expression for $\breve{S}_{-i}{}''(p)$ can be derived from PABA FOC. The equality in (4) is valid for an interval of prices. Thus:

$$G\left[\breve{S}(p)\right] - \breve{S}_{-i}{}'(p)\left[p - C'\left(\breve{S}_i(p)\right)\right] \equiv 0.$$

Differentiating both sides with respect to p and algebraic manipulations yields:

$$-\breve{S}_{-i}{}''(p)\left[p - C'\left(\breve{S}_i(p)\right)\right] \equiv \breve{S}_{-i}{}'(p)\left[1 - C''\left(\breve{S}_i(p)\right)\breve{S}_i{}'(p)\right] - G'\left[\breve{S}(p)\right]\breve{S}'(p). \tag{10}$$

Hence, (9) can be written

[9] It is never profitable to offer a unit below $\breve{p}(0)$, as the unit is always accepted at this price.

$$\frac{\partial H}{\partial p} = G'\left[\breve{S}_{-i}(p)+S_i\right]\left[\breve{S}'_{-i}(p)+\breve{S}_{-i}{}'(p)\left[1-C''\left(\breve{S}_i(p)\right)\breve{S}_i{}'(p)\right]-G'\left[\breve{S}(p)\right]\breve{S}'(p)-\breve{S}_{-i}{}'(p)= \\ = G'\left[\breve{S}_{-i}(p)+S_i\right]\breve{S}'_{-i}(p)-\breve{S}_{-i}{}'(p)C''\left(\breve{S}_i(p)\right)\breve{S}_i{}'(p)-G'\left[\breve{S}(p)\right]\breve{S}'(p). \quad (11)$$

Recall that $C'' \geq 0$. Thus for $G'(x)=const>0$, $\frac{\partial H}{\partial p}<0$ is satisfied for $p \in \left[\breve{p}(0),\overline{p}\right]$ and $S_i \in \left[0, \overline{\varepsilon}/N\right]$, as $0<\breve{S}'_{-i}(p)<\breve{S}'(p)$[10]. Thus $G'(x)=const>0$, ensures that (6) is a SFE.

Note that the weaker condition $G'(x)\geq 0$ is necessary to ensure that expected profits are locally maximized for all production units, but this is not sufficient for a SFE.

There are also cases, where a symmetric SFE can be ruled out. Assume that there is some S_i, for which $C''(S_i)=0$ and $G'(NS_i)<0$. It is straightforward to show that $f'(NS_i)\geq 0$ is a sufficient condition for the latter. Then, according to (11), $\left.\frac{\partial H}{\partial p}\right|_{p=p^*}>0$ for this S_i. In this case, $p=p^*$ locally minimizes the expected profit from a unit—the marginal unit when firm i is supplying S_i units of power—and there will be a profitable deviation. Thus if there is some ε, for which $C''(\varepsilon/N)=0$ and $f'(\varepsilon)\geq 0$, then a smooth symmetric SFE can be ruled out.

There is some intuition behind the non-existent equilibrium. In case of a monopolist or Cournot player, a similar problem occurs when the demand or residual demand is sufficiently convex [10]. In the PABA, it follows from (7) that $1-F\left[\breve{S}_{-i}(p)+S_i\right]$ can be interpreted as the residual demand of the marginal unit, when firm i is supplying S_i units of power[11]. It follows from PABA FOC that $\breve{S}_{-i}(p)$ is concave for a sufficiently high $f'(\varepsilon)$, see (10). If it is sufficiently concave, the residual demand becomes convex enough to rule out equilibria. The existence of equilibria is easier to guarantee in UPA, as SFE of UPA are independent of $f(\varepsilon)$. Symmetric SFE of UPA exists as long as the demand function is concave [17].

3.4. The inverse polynomial probability distribution

Symmetric SFE is guaranteed for the class of probability distributions satisfying $G'(x)=const>0$, or equivalently when

[10] Recall that supply functions are assumed to be monotonically increasing. This also follows from PABA FOC.
[11] Previously, Burlow and Klemperer have noted that the probability that a bid is accepted, i.e. 1-F, can be interpreted as the residual demand [4].

$$G(x) = \frac{1 - F(x)}{f(x)} = \alpha x + \beta, \tag{12}$$

where $\alpha, \beta > 0$[12]. The differential equation is straightforward to solve, after writing it on the form:

$$\frac{f(x)}{1 - F(x)} = \frac{1}{\alpha x + \beta}.$$

With the boundary condition $F(0) = 0$, the solution is:

$$F(x) = 1 - \beta^{\frac{1}{\alpha}} (\alpha x + \beta)^{\frac{-1}{\alpha}}. \tag{13}$$

Hence,

$$f(x) = \beta^{\frac{1}{\alpha}} (\alpha x + \beta)^{\frac{-1}{\alpha} - 1}. \tag{14}$$

This is a generalized version of the Zipfian distribution [24]. As illustrated in Fig. 1, the parameter β determines $f(0)$. Further, in case of a large α, f has a steep negative slope for small arguments and a thick tail for large arguments. It is the other way around for small α. The density function is decreasing and strictly convex for all $\alpha, \beta > 0$. In the balancing market, small imbalances are more likely than large imbalances. Thus the inverse polynomial probability distribution catches an important characteristic of this market.

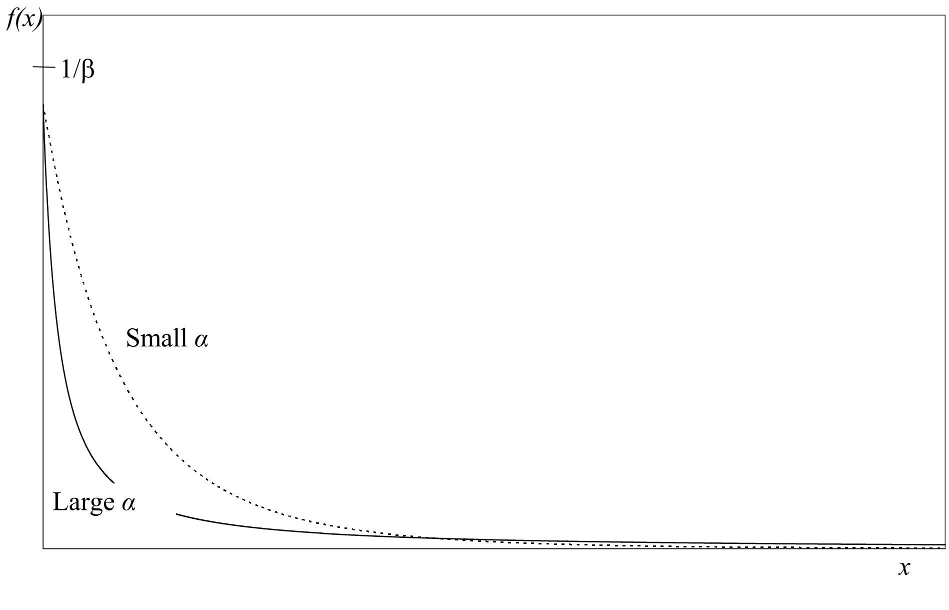

Fig. 1. *The influence of α and β on the probability density function f(x).*

With the inverse polynomial probability distribution, the PABA FOC can be simplified to:

$$\alpha N S_i(p) + \beta - (N-1) S_i'(p)(p - C'(S_i(p))) \equiv 0,$$

[12] $\alpha > 0$, as $G' > 0$ and $\beta > 0$, as $G(0) > 0$. The latter as $f(\cdot)$ has been assumed to be continuously differentiable.

which is similar to the UPA FOC [17]:

$$S_i(p)-(N-1)S_i'(p)(p-C'(S_i(p)))\equiv 0.$$

It follows from (6) that the equilibrium marginal bid in a PABA with the inverse polynomial probability distribution is:

$$p(\varepsilon)=\frac{N\left(\alpha\bar{\varepsilon}+\beta\right)^{\frac{1-N}{\alpha N}}\bar{p}+\int_{\varepsilon}^{\bar{\varepsilon}}(N-1)C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}du}{N(\alpha\varepsilon+\beta)^{\frac{1-N}{\alpha N}}}. \quad (15)$$

4. Comparing pay-as-bid and uniform-price auctions

Demand is assumed to be inelastic. Thus total production is the same in a pay-as-bid and uniform-price auction. Further, only symmetric equilibria are considered. This means that for every demand outcome, the most cost-effective generators will be accepted in both auctions. Hence, production costs are the same in both auctions for all outcomes. Average prices and mark-ups will differ, however. This difference is investigated by comparing firms' total expected revenue in the two auctions. To ensure a SFE in both auctions, it is assumed that demand follows the inverse polynomial probability distribution. It is first shown that revenues in PABA are lower or equal to revenues in UPA, in case of constant marginal costs. This result can then be used to prove the same inequality for non-decreasing marginal costs.

4.1. Constant marginal costs

It follows from (1) that the total expected revenue for all firms in a PABA is:

$$R_p=\int_0^{\bar{\varepsilon}}(1-F(S))p(S)dS=\int_0^{\bar{\varepsilon}}(1-F(\varepsilon))p(\varepsilon)d\varepsilon.$$

By means of (13) and (15) it can be shown that:

$$R_P=\int_0^{\bar{\varepsilon}}\frac{\beta^{\frac{1}{\alpha}}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}}\left\{\bar{p}N\left(\alpha\bar{\varepsilon}+\beta\right)^{\frac{1-N}{\alpha N}}+(N-1)\int_{\varepsilon}^{\bar{\varepsilon}}C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}du\right\}}{N(\alpha\varepsilon+\beta)^{\frac{1-N}{\alpha N}}}d\varepsilon. \quad (16)$$

Constant marginal costs are assumed in this section, i.e. $C'(u/N)\equiv c.$ For this case, it can be shown by straightforward integration and the substitution $t=\frac{\alpha\varepsilon}{\beta}$ that:

$$R_P = \frac{\beta^{\frac{N-1}{\alpha N}}\left(\bar{p}-c\right)}{\left(\alpha\bar{\varepsilon}+\beta\right)^{\frac{N-1}{\alpha N}}}\int_0^{\bar{\varepsilon}}\left(1+\frac{\alpha\varepsilon}{\beta}\right)^{\frac{-1}{\alpha N}} d\varepsilon + \beta^{\frac{1}{\alpha}} c\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}} d\varepsilon = \tag{17}$$
$$= \left(\bar{p}-c\right)\bar{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}} c\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}} d\varepsilon,$$

where

$$g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) = \frac{\int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}}\left(1+t\right)^{\frac{-1}{\alpha N}} dt}{\frac{\alpha\bar{\varepsilon}}{\beta}\left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{N-1}{\alpha N}}} = \frac{\left(1+\frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha N}+1} - 1}{\left(1-\frac{1}{\alpha N}\right)\frac{\alpha\bar{\varepsilon}}{\beta}\left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{N-1}{\alpha N}}}. \tag{18}$$

The following can be shown by means of integration by parts:

$$\beta^{\frac{1}{\alpha}}\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}} d\varepsilon = \beta^{\frac{1}{\alpha}}\left[\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}}\varepsilon\right]_0^{\bar{\varepsilon}} + \beta^{\frac{1}{\alpha}}\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}-1}\varepsilon d\varepsilon = \left[1-F\left(\bar{\varepsilon}\right)\right] + \int_0^{\bar{\varepsilon}} f\left(\varepsilon\right)\varepsilon d\varepsilon.$$

Thus $\beta^{\frac{1}{\alpha}} c\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}} d\varepsilon$ is the expected production cost and $\left(\bar{p}-c\right)\bar{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)$, see (17), is a measure of the mark-up.

The equilibrium marginal bid for symmetric firms in a UPA is [13]:

$$p_U\left(\varepsilon\right) = \frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + \left(N-1\right)\varepsilon^{N-1}\int_{\varepsilon}^{\bar{\varepsilon}}\frac{C'\left(u/N\right)du}{u^N}, \quad \text{if } \varepsilon \geq 0. \tag{19}$$

The total expected revenue for firms in a uniform-price auction is:

$$R_U = \int_0^{\bar{\varepsilon}} f\left(\varepsilon\right)\varepsilon p_U\left(\varepsilon\right)d\varepsilon + \left(1-F\left(\bar{\varepsilon}\right)\right)\bar{\varepsilon}\bar{p}.$$

The second term is the contribution from demand outcomes exceeding market capacity. Assuming constant marginal costs, it can be shown by means of (13), (14) and (19) that:

$$R_U = \int_0^{\bar{\varepsilon}}\beta^{\frac{1}{\alpha}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}-1}\varepsilon\left[\frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + \left(N-1\right)\varepsilon^{N-1}\int_{\varepsilon}^{\bar{\varepsilon}}\frac{cdu}{u^N}\right]d\varepsilon + \beta^{\frac{1}{\alpha}}\left(\alpha\bar{\varepsilon}+\beta\right)^{\frac{-1}{\alpha}}\bar{\varepsilon}\bar{p}.$$

By means of integration by parts, the expression can be simplified to:

$$R_U = \frac{N\left(\bar{p}-c\right)}{\bar{\varepsilon}^{N-1}}\int_0^{\bar{\varepsilon}}\left(\frac{\alpha\varepsilon}{\beta}+1\right)^{\frac{-1}{\alpha}}\varepsilon^{N-1}\,\mathrm{d}\varepsilon + \beta^{\frac{1}{\alpha}} c\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}} d\varepsilon = \tag{20}$$
$$= \left(\bar{p}-c\right)\bar{\varepsilon} g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}} c\int_0^{\bar{\varepsilon}}\left(\alpha\varepsilon+\beta\right)^{\frac{-1}{\alpha}} d\varepsilon,$$

where

$$g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) = \frac{N}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} \int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}} (1+t)^{\frac{-1}{\alpha}} t^{N-1}\, \mathrm{d}t. \tag{21}$$

The integral can be solved by repeated use of integration by parts.

From (17) and (20) it follows that:

$$R_U - R_P = (\bar{p} - c)\bar{\varepsilon}\left[g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)\right].$$

The contour plot of $\frac{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}$ in Fig. 2, shows the relative decrease of mark-ups when switching from a UPA to a PABA. The plot is not so sensitive to the number of firms. As the ratio is positive in a wide parameter range, it seems that $R_U - R_P \geq 0.$ This inequality can be proven mathematically [12].

Theorem 1. If $f(x) = \beta^{\frac{1}{\alpha}}(\alpha x + \beta)^{\frac{-1}{\alpha}-1},\ \alpha > 0, \frac{\alpha\bar{\varepsilon}}{\beta} \geq 0, N \geq 2,$ and marginal costs are constant, then the expected revenue of symmetric firms in a Pay-As-Bid procurement auction is weakly lower than their expected revenue in a Uniform-Price procurement auction.

Fig. 2 shows that switching from a UPA to a PABA all but wipes out mark-ups, if $\bar{\varepsilon} >> \beta$ and $\alpha < 1.$ As can be seen in Fig. 3, these parameter values correspond to a very low risk of power shortage. On the other hand, mark-ups are nearly unchanged for either very large α (fat tail of probability density function) or very small $\bar{\varepsilon}$ (small capacity), which both imply a very high risk of power shortage, see Fig. 3.

In most electric power markets, it is reasonable to assume that power shortages occur in the range from once every hundred years to 100 times per year. It roughly corresponds to the probability for a power shortage during an hour—a normal length of the delivery period—being 0.01-10^{-6}. By comparing Fig. 3 with a contour plot as in Fig. 2, it can be deduced that switching to a pay-as-bid auction in an electric power market can reduce average mark-ups by 60-99%, if $\alpha < 0.1$ and N=2; the lower the risk of power shortage, the larger the impact. One can show that the impact is somewhat reduced, if the number of symmetric firms increase. For

N=10 and $\alpha < 0.1$, switching to a pay-as-bid auction would reduce average mark-ups by 20-90%. On the other hand for $\alpha > 1$, which corresponds to a more convex probability density function, there is — independent of the number of symmetric firms — hardly any gain in switching to a pay-as-bid auction in the electric power market.

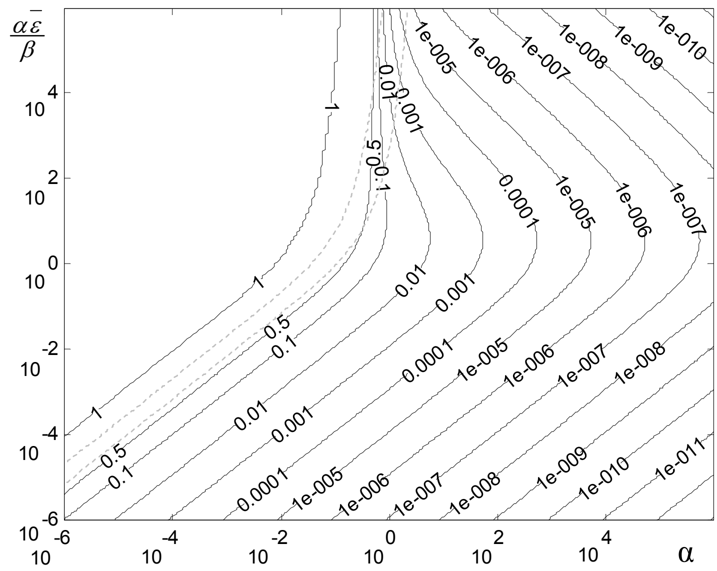

Fig. 2. *Contour plot of* $\dfrac{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}$ *when N=2. The gray dotted line indicates a region with a risk of power shortage realistic for electric power markets.*

There is some intuition behind the importance of *α* in the comparison of PABA and UPA. Equilibrium bids in UPA are not influenced by the probability distribution of demand [13,17]. Bids in PABA are, however, sensitive to α. A smaller α makes low demand outcomes more likely. Intuitively, this increases the elasticity of residual demand for small S_i and accordingly mark-ups are lower for these units in a PABA[13]. Thus, in case of smaller *α*, there are two effects in the PABA that drive down firms' expected revenues, lower mark-ups for low demand outcomes and an increased probability of low demand outcomes. In the UPA, it is only the latter effect that drives down firms expected revenues. The same intuition may also explain why firms' expected revenues are lower in PABA compared to UPA for decreasing

[13] Recall that $1 - F\left[\breve{S}_{-i}(p) + S_i\right]$ can be interpreted as the residual demand of the marginal unit when firm *i* is supplying S_i units of power.

probability densities. One might expect different results for increasing probability densities, if SFE of PABA exists in this case.

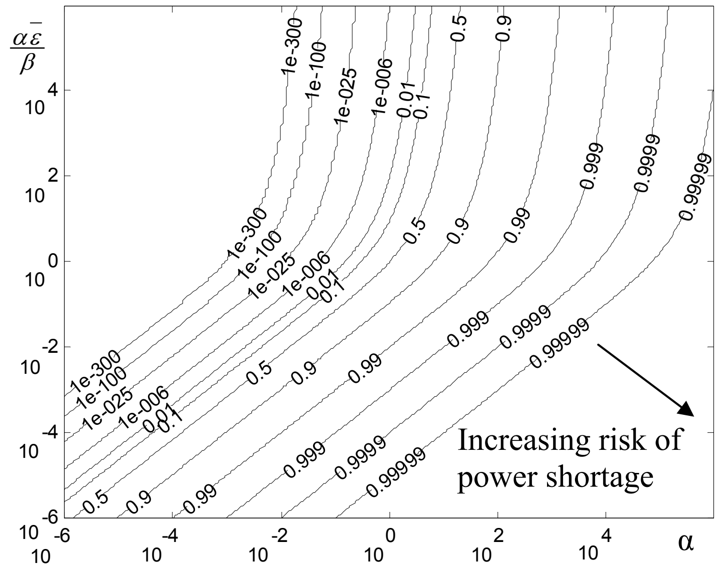

Fig. 3. *Contour plot of 1-* $F(\bar{\varepsilon}) = \beta^{\frac{1}{\alpha}}\left(\alpha\bar{\varepsilon}+\beta\right)^{\frac{-1}{\alpha}}$, *i.e. the probability of power shortage.*

4.2 Non-decreasing marginal costs

In Section 4.1 it was shown that switching from a UPA to PABA reduces firms' revenues, if marginal costs are constant and the demand follows an inverse polynomial probability distribution. Using Theorem 1, this section will show that the conclusion can be generalized to non-decreasing marginal costs.

The revenue contribution from the term related to the price cap in (16) does not depend on marginal costs. Thus it follows from (16) and (17) that:

$$R_P = \bar{p}\bar{\varepsilon}g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \int_0^{\bar{\varepsilon}} \frac{\beta^{\frac{1}{\alpha}}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}}(N-1)\int_{\varepsilon}^{\bar{\varepsilon}} C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}du}{N(\alpha\varepsilon+\beta)^{\frac{1-N}{\alpha N}}}d\varepsilon.$$

By reversing the order of integration, it can be shown that [22]:

$$R_p = \overline{p\varepsilon}g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \frac{\beta^{\frac{1}{\alpha}}(N-1)}{N}\int_0^{\bar{\varepsilon}} C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}\int_0^u (\alpha\varepsilon+\beta)^{\frac{-1}{\alpha N}}\, d\varepsilon du =$$

$$= \overline{p\varepsilon}g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + (N-1)\int_0^{\bar{\varepsilon}} C'(u/N)\underbrace{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{1-N}{\alpha N}-1}\frac{\left(\frac{\alpha u}{\beta}+1\right)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1}}_{h_P\left(\alpha, N, \frac{\alpha u}{\beta}\right)} du. \tag{22}$$

Similarly it follows from (19) and (20) that:

$$R_U = \overline{p\varepsilon}g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}}(N-1)\int_0^{\bar{\varepsilon}} (\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}-1}\varepsilon^N \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(u/N)du}{u^N}\, d\varepsilon =$$

$$= \overline{p\varepsilon}g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}}(N-1)\int_0^{\bar{\varepsilon}} \frac{C'(u/N)}{u^N}\int_0^u (\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}-1}\varepsilon^N\, d\varepsilon du = \tag{23}$$

$$= \overline{p\varepsilon}g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + (N-1)\int_0^{\bar{\varepsilon}} C'(u/N)\underbrace{\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}} \frac{(t+1)^{\frac{-1}{\alpha}-1}t^N}{\alpha}\, dt}_{h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)} du.$$

The integral in $h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)$ can be solved analytically by means of repeated use of integration by parts.

Let $h_{\alpha N}(x) = h_U(\alpha, N, x) - h_P(\alpha, N, x)$. (22) and (23) now imply that:

$$\Delta R = R_U - R_P = \overline{p\varepsilon}\left[g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)\right] + (N-1)\int_0^{\bar{\varepsilon}} C'(u/N)h_{\alpha N}\left(\frac{\alpha u}{\beta}\right)du. \tag{24}$$

Fig. 4 presents a contour plot of $h_{\alpha N}(x)$. The levels in the contour plot are very sensitive to *N,* but not the pattern. It seems that $h_{\alpha N}(x)$ has profile -|+ for $N \geq 2$ and $x \geq 0$. This is verified analytically in [12]. If $h_{\alpha N}\left(\frac{\alpha u}{\beta}\right)$ changes sign for $u < \bar{\varepsilon}$, let u^* be this point, otherwise set $u^* = \bar{\varepsilon}$. Use (24) to calculate ΔR_1 for the non-decreasing cost function $C_1(\varepsilon/N)$. Next calculate ΔR_2 for the constant marginal cost $c_2 = C_1'(u^*/N)$. Compared to $C_1'(\varepsilon/N)$, c_2 puts a (weakly) higher weight on negative $h_{\alpha N}(x)$ and a (weakly) lower weight on positive $h_{\alpha N}(x)$. Thus

$$\Delta R_1 \geq \Delta R_2.$$

From Theorem 1 it follows that $\Delta R_2 \geq 0$. Thus $\Delta R_1 \geq 0$ and $R_U \geq R_P$ is true also for non-decreasing marginal costs.

Theorem 2. If $f(x)=\beta^{\frac{1}{\alpha}}(\alpha x+\beta)^{\frac{-1}{\alpha}-1}$, $\alpha>0, \frac{\alpha\bar{\varepsilon}}{\beta}\geq 0, N\geq 2,$ and marginal costs are non-decreasing, then the expected revenue of symmetric firms in a Pay-As-Bid procurement auction is weakly lower than their expected revenue in a Uniform-Price procurement auction.

Proof: See [12].

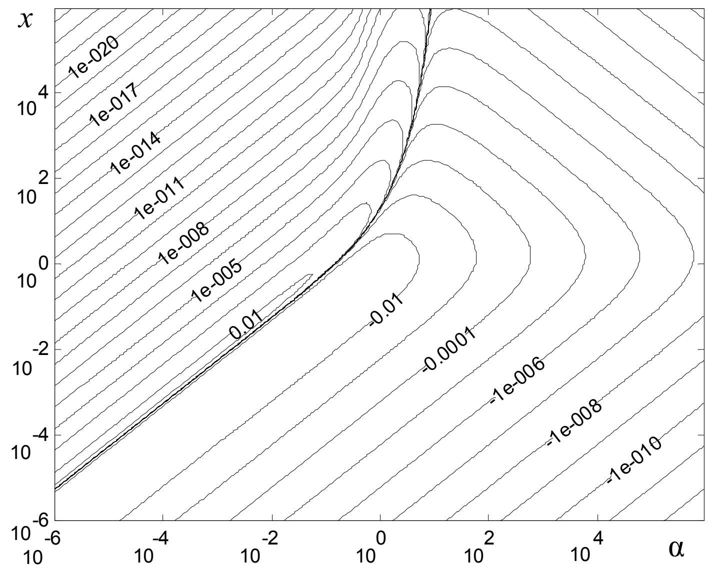

Fig. 4. *Contour plot of* $h_{\alpha N}(x)$ for N=2.

It is obvious that R_U=R_P=0 when $\bar{\varepsilon}=0$, i.e. the market capacity is zero. It can also be shown that firms' total expected revenues are the same in both auctions under perfect competition and monopoly.

Theorem 3. If $f(x)=\beta^{\frac{1}{\alpha}}(\alpha x+\beta)^{\frac{-1}{\alpha}-1}$, $\alpha>0, \frac{\alpha\bar{\varepsilon}}{\beta}\geq 0,$ marginal costs are non-decreasing and $N\rightarrow\infty$ or N=1, then the expected revenue of symmetric firms in a Pay-As-Bid auction is identical to their expected revenue in a Uniform-Price auction.

Proof: See Appendix.

Recall that demand is assumed to be inelastic and accordingly independent of the auction design. Thus Theorem 2 implies that the demand-weighted average price is weakly lower in

PABA than in UPA. Further, only symmetric equilibria are considered. This means that for every demand outcome, the most cost-effective generators will be accepted in both auctions. Thus production costs are the same in both procurement auctions, and average mark-ups are weakly lower in PABA compared to UPA.

5. EXAMPLE

Assume $N=2$, $\alpha=1$, and linear marginal costs, i.e. $C'(x) \equiv \gamma x$. The marginal bid in the PABA for these parameter values can be calculated by means of integration by parts and (15).

$$\frac{p(\varepsilon)}{\beta} = \frac{\left(\frac{2\bar{p}}{\beta} + \frac{\gamma\bar{\varepsilon}}{\beta} + 2\gamma\right)\left(\frac{\bar{\varepsilon}}{\beta} + 1\right)^{-1/2}}{2\left(\frac{\varepsilon}{\beta} + 1\right)^{-1/2}} + \frac{\gamma\varepsilon}{2\beta} - \gamma\left(\frac{\varepsilon}{\beta} + 1\right)$$

The demand and price are normalized with respect to β. In the PABA, the average price as a function of demand, also called equilibrium price, is:

$$\frac{\hat{p}(\varepsilon)}{\beta} = \frac{\int_0^{\varepsilon} p(x)dx}{\beta\varepsilon} = \left(\frac{2\bar{p}}{\beta} + \frac{\gamma\bar{\varepsilon}}{\beta} + 2\gamma\right)\left(\frac{\bar{\varepsilon}}{\beta} + 1\right)^{-1/2} \frac{\left(\frac{\varepsilon}{\beta} + 1\right)^{3/2} - 1}{3\varepsilon/\beta} + \frac{\gamma\varepsilon}{4\beta} - \gamma\left(\frac{\varepsilon}{2\beta} + 1\right).$$

The equilibrium price in the UPA can be calculated by means of (19).

$$p_U(\varepsilon) = \frac{\bar{p}\varepsilon}{\bar{\varepsilon}} + \frac{\varepsilon\gamma}{2}\ln\left(\frac{\bar{\varepsilon}}{\varepsilon}\right) = \beta\frac{\bar{p}\varepsilon/\beta^2}{\bar{\varepsilon}/\beta} + \beta\frac{\varepsilon\gamma/\beta}{2}\ln\left(\frac{\bar{\varepsilon}/\beta}{\varepsilon/\beta}\right).$$

Fig. 5 shows $p_U(\varepsilon)$, $p(\varepsilon)$, and $\hat{p}(\varepsilon)$ for $\frac{\bar{p}}{\beta} = 10^3$ and $\frac{\bar{\varepsilon}}{\beta} = 10^4$. The latter corresponds to a risk of power shortage $\approx 10^{-4}$. The equilibrium price in the uniform-price auction equals $C'(0)=0$ at zero demand. The unit with the lowest marginal cost, the cheapest unit, still contributes to profits, as it is paid the marginal bid for $\varepsilon>0$. It is generally true that the lowest bid in the pay-as-bid auction is higher than $C'(0)$. Otherwise the cheapest unit would not contribute to profits, as accepted bids are always paid there bid. Thus the equilibrium price is higher in the PABA when demand is sufficiently small. In both procurement auctions, all units, except for the unit with the highest bid, are offered below the price cap. Thus the equilibrium price of PABA is always below the price cap. In the uniform-price auction, on the other hand, the equilibrium price equals the price cap when demand equals or exceeds the market capacity. Thus the equilibrium price is lower in the PABA when demand is sufficiently high.

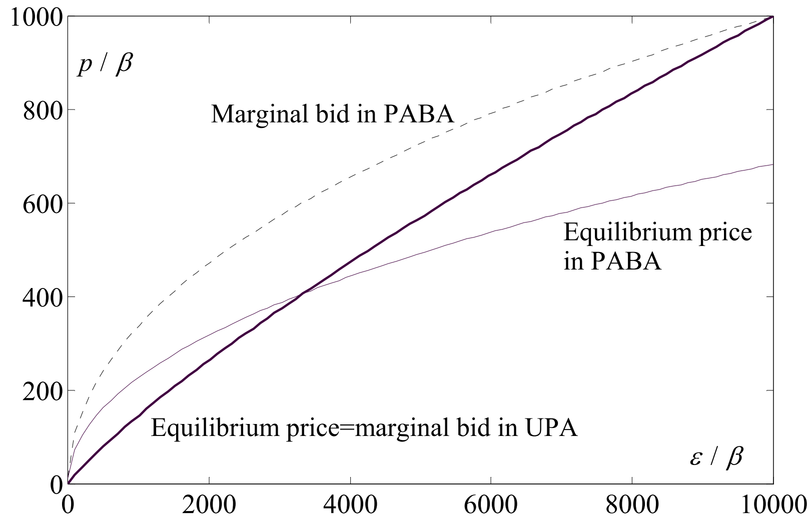

Fig. 5. *Example for duopoly: prices as a function of demand (ε) are compared for the uniform-price auction (UPA) and pay-as-bid auction (PABA).*

6. CONCLUSIONS

A Supply Function Equilibrium (SFE) has been derived for a pay-as-bid auction (PABA) of an electric power market, where demand is inelastic and firms are symmetric. Further, demand exceeds the market capacity with a positive probability, which could be arbitrarily small. This assumption rules out multiple equilibria. Unlike SFE of uniform-price auctions (UPA), pure strategy equilibria of PABA do not always exist. In particular, it can be shown that SFE of PABA do not exist if there is an interval where the probability density of the demand is non-decreasing and marginal costs are constant. However, if demand is given by an inverse polynomial probability distribution, for which the inverse of the hazard rate is linear, then a SFE of PABA always exists. The probability density function is strictly convex and monotonically decreasing.

The derived equilibrium of a pay-as-bid procurement auction is compared to a SFE of the UPA. For the inverse polynomial probability distribution, it can be shown that the demand-weighted average price in the PABA is equal to or lower than in the UPA. Equality occurs in case of a monopoly or perfect competition. An analogous calculation would show that the demand-weighted average price in the Pay-As-Bid sales auction is (weakly) higher compared to a uniform-price sales auction. Thus producer profits in the balancing market should be reduced after a switch to PABA. For a probability density function with a low degree of convexity, switching from a UPA to a PABA will drastically reduce the average mark-up in electricity procurement auctions. With a high degree of convexity, the change in the average mark-up is negligible. That mark-ups are lower and consumer surplus higher in PABA is in line with previous theoretical studies [8,9], even if they are based on other assumptions. The result contradicts the findings in an experimental study [20], but the experimental result might have been different with uncertain demand.

The equilibrium price — the average price as a function of demand— is higher in PABA compared to UPA when demand is sufficiently low, but lower, when demand is sufficiently high. This seems to be in agreement with the experimental finding that the price volatility is lower in PABA compared to UPA [20].

It is assumed that firms are risk-neutral. Risk aversion does not change the SFE of a UPA, as firms get the best price for every demand outcome, given bids of the competitors. A risk-averse firm in a PABA, however, would put less weight on high demand outcomes, where profits are high, and more weight on low demand outcomes, where profits are low. Hence, given the bids of the competitors, risk-averse firms decrease their bids to increase profits for low-demand outcomes. Intuitively this would be true also in equilibrium. The intuitive result can be proven for private value bidders in a procurement auction with single objects [19]. Thus it seems that, with risk-averse bidders, the advantages with PABA would increase. Another advantage with PABA is that the risk for tacit collusion is lower in this auction compared to UPA. This is shown by both Fabra [6] and Klemperer [18].

The larger risk for non-existent pure strategy equilibria in the PABA, which is shown in this paper and by Fabra et al. [9], could be a disadvantage. Kahn et al. [16] have pointed out another disadvantage. In a UPA, it is optimal for small firms to simply bid their marginal costs. In PABA, however, all firms will be forced to spend money on forecasting market prices, if they are to receive any contributions to profits. This will introduce an additional fixed cost for small firms, which would be disadvantageous to competition in the long-run.

As small imbalances are more likely than large imbalances, the inverse polynomial probability distribution is a more reasonable representation of the uncertain imbalance in balancing market compared to the uniform distribution used by e.g. Federico & Rahman [8]. Still an interesting future topic would be to prove that SFE of PABA exists for even more general probability distributions and to compare PABA and UPA for these distributions.

In this paper, the SFE of PABA is derived for symmetric firms. Analogous to [14], it should be possible to analytically derive SFE of PABA for firms with identical constant marginal costs, but asymmetric capacities; at least for the inverse polynomial probability distribution. For more general cost functions and other probability distributions, asymmetric SFE could be calculated numerically as in [15].

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APPENDIX

It is known from (20) and (22) that:

$$R_U - R_P = \overline{p}\overline{\varepsilon}\left[g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right)\right] +$$

$$+ (N-1)\int_0^{\overline{\varepsilon}} C'(u/N)\left[h_U\left(\alpha, N, \tfrac{\alpha u}{\beta}\right) - h_P\left(\alpha, N, \tfrac{\alpha u}{\beta}\right)\right]dtdu.$$

Thus the auctions have the same expected revenue under perfect competition, if

$\lim_{N\to\infty} g_U\left(\alpha, N, \tfrac{\alpha\overline{\varepsilon}}{\beta}\right) = \lim_{N\to\infty} g_P\left(\alpha, N, \tfrac{\alpha\overline{\varepsilon}}{\beta}\right)$ and $\lim_{N\to\infty} (N-1)h_U\left(\alpha, N, \tfrac{\alpha\overline{\varepsilon}}{\beta}\right) = \lim_{N\to\infty} (N-1)h_P\left(\alpha, N, \tfrac{\alpha\overline{\varepsilon}}{\beta}\right)$.

These two equalities are proven below.

It follows from (18) that:

$$\lim_{N\to\infty} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) = \lim_{N\to\infty} \frac{\left(1+\frac{\alpha\overline{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha N}+1} - 1}{\left(1-\frac{1}{\alpha N}\right)\frac{\alpha\overline{\varepsilon}}{\beta}\left(\frac{\alpha\overline{\varepsilon}}{\beta}+1\right)^{\frac{N-1}{\alpha N}}} = \frac{1}{\left(\frac{\alpha\overline{\varepsilon}}{\beta}+1\right)^{\frac{1}{\alpha}}}.$$

As $\lim_{N\to\infty} \frac{Nt^{N-1}}{\left(\frac{\alpha\overline{\varepsilon}}{\beta}\right)^N} = 0$, if $t < \frac{\alpha\overline{\varepsilon}}{\beta}$, it can be shown by means of (21) that:

$$\lim_{N\to\infty} g_U\left(\alpha, N, \tfrac{\alpha\bar{\varepsilon}}{\beta}\right) = \lim_{N\to\infty} \frac{N\int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}} (1+t)^{\frac{-1}{\alpha}} t^{N-1} dt}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} = \lim_{N\to\infty} \frac{N\left(1+\frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha}}}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} \int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}} t^{N-1} dt = \left(1+\tfrac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha}} =$$

$$= \lim_{N\to\infty} g_P\left(\alpha, N, \tfrac{\alpha\bar{\varepsilon}}{\beta}\right).$$

Using (22) it can be shown that:

$$\lim_{N\to\infty}(N-1)h_P\left(\alpha, N, \frac{\alpha u}{\beta}\right) = \lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}+1\right)^{\frac{1-N}{\alpha N}-1} \frac{\left(\frac{\alpha u}{\beta}+1\right)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1} = \frac{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1} u}{\beta}.$$

As $\lim_{N\to\infty} \frac{(N-1)t^N}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} = 0, \text{if } t < \frac{\alpha\bar{\varepsilon}}{\beta}$, it follows from (21) that:

$$\lim_{N\to\infty}(N-1)h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right) = \lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}\right)^{-N} \int_0^{\frac{\alpha u}{\beta}} \frac{(t+1)^{\frac{-1}{\alpha}-1} t^N dt}{\alpha} =$$

$$= \lim_{N\to\infty}(N-1)\left(\tfrac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}\left(\frac{\alpha u}{\beta}\right)^{-N} \int_0^{\frac{\alpha u}{\beta}} \frac{t^N dt}{\alpha} = \frac{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1} u}{\beta} = \lim_{N\to\infty}(N-1)h_p\left(\alpha, N, \frac{\alpha u}{\beta}\right).$$

Monopoly, i.e. N=1, has not been considered in the analysis, as it has been assumed that $N\geq 2$. In this case, the auctioneer must buy from the monopolist, and the monopolist would offer all of his units at (or arbitrarily close to[14]) the price cap in both auctions. Thus the expected revenue is identical in the two auctions.

[14] As supply functions are assumed to fulfil $S'(p) < \infty$.

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