Chapter 4

Topics in Consumer Theory

4.1 Homothetic and Quasilinear Utility Functions

One of the chief activities of economics is to try to recover a consumer's preferences over all bundles from observations of preferences over a few bundles. If you could ask the consumer an infinite number of times, "Do you prefer x to y?", using a large number of different bundles, you could do a pretty good job of figuring out the consumer's indifference sets, which reveals her preferences. However, the problem with this is that it is impossible to ask the question an infinite number of times.[1] In doing economics, this problem manifests itself in the fact that you often only have a limited number of data points describing consumer behavior.

One way that we could help make the data we have go farther would be if observations we made about one particular indifference curve could help us understand all indifference curves. There are a couple of different restrictions we can impose on preferences that allow us to do this.

The first restriction is called **homotheticity**. A preference relation is said to be homothetic if the slope of indifference curves remains constant along any ray from the origin. Figure 4.1 depicts such indifference curves.

If preferences take this form, then knowing the shape of one indifference curve tells you the shape of all indifference curves, since they are "radial blowups" of each other. Formally, we say a preference relation is **homothetic** if for any two bundles x and y such that $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha > 0$.

We can extend the definition of homothetic preferences to utility functions. A continuous

[1] In fact, to completely determine the indifference sets you would have to ask an uncountably infinite number of questions, which is even harder.

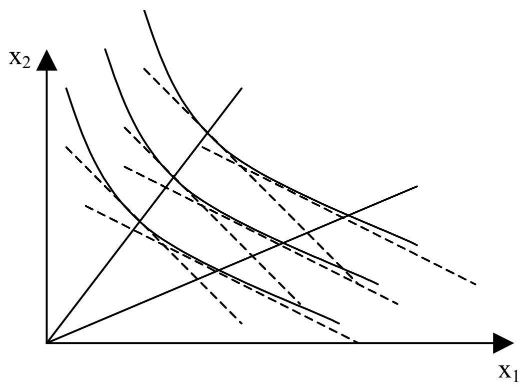

Figure 4.1: Homothetic Preferences

preference relation $\succeq$ is homothetic if and only if it can be represented by a utility function that is homogeneous of degree one. In other words, homothetic preferences can be represented by a function $u()$ that such that $u(\alpha x) = \alpha u(x)$ for all x and $\alpha > 0$. Note that the definition does not say that every utility function that represents the preferences must be homogeneous of degree one – only that there must be at least one utility function that represents those preferences and is homogeneous of degree one.

EXAMPLE: Cobb-Douglas Utility: A famous example of a homothetic utility function is the Cobb-Douglas utility function (here in two dimensions):

$$u(x_1, x_2) = x_1^a x_2^{1-a} : a > 0.$$

The demand functions for this utility function are given by:

$$\begin{aligned} x_1(p, w) &= \frac{aw}{p_1} \\ x_2(p, w) &= \frac{(1-a) w}{p_2}. \end{aligned}$$

Notice that the ratio of x_1 to x_2 does not depend on w. This implies that Engle curves (wealth expansion paths) are straight lines (see MWG pp. 24-25). The indirect utility function is given by:

$$v(p, w) = \left(\frac{aw}{p_1}\right)^a \left(\frac{(1-a) w}{p_2}\right)^{1-a} = w \left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a}.$$

Another restriction on preferences that can allow us to draw inferences about all indifference curves from a single curve is called **quasilinearity.** A preference relation is quasilinear if there is one commodity, called the numeraire, which shifts the indifference curves outward as consumption of it increases, without changing their slope. Indifference curves for quasilinear preferences are illustrated in Figure 3.B.6 of MWG.

Again, we can extend this definition to utility functions. A continuous preference relation is quasilinear in commodity 1 if there is a utility function that represents it in the form $u(x) = x_1 + v(x_2, ..., x_L)$.

EXAMPLE: Quasilinear utility functions take the form $u(x) = x_1 + v(x_2, ..., x_L)$. Since we typically want utility to be quasiconcave, the function $v()$ is usually a concave function such as $\log x$ or $\sqrt{x}$. So, consider:

$$u(x) = x_1 + \sqrt{x_2}.$$

The demand functions associated with this utility function are found by solving:

$$\begin{aligned} & \max x_1 + x_2^{0.5} \\ s.t. \quad : \quad & p \cdot x \leq w \end{aligned}$$

or, since $x_1 = -x_2 \frac{p_2}{p_1} + \frac{w}{p_1}$,

$$\max -x_2 \frac{p_2}{p_1} + \frac{w}{p_1} + x_2^{0.5}.$$

The associated demand curves are

$$\begin{aligned} x_1(p, w) &= -\frac{1}{4}\frac{p_1}{p_2} + \frac{w}{p_1} \\ x_2(p, w) &= \left(\frac{p_1}{2p_2}\right)^2 \end{aligned}$$

and indirect utility function:

$$v(p, w) = \frac{1}{4}\frac{p_1}{p_2} + \frac{w}{p_1}.$$

Isoquants of this utility function are drawn in Figure 4.2.

4.2 Aggregation

Our previous work has been concerned with developing the testable implications of the theory of the consumer behavior on the individual level. However, in any particular market there are large numbers of consumers. In addition, often in empirical work it will be difficult or impossible to collect data on the individual level. All that can be observed are aggregates: aggregate consumption of the various commodities and a measure of aggregate wealth (such as GNP). This raises the

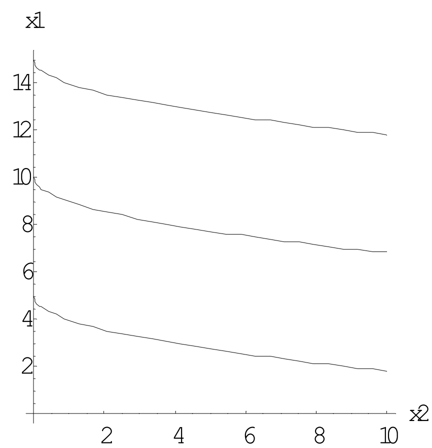

Figure 4.2: Quasilinear Preferences

natural question of whether or not the implications of individual demand theory also apply to aggregate demand.

To make things a little more concrete, suppose there are N consumers numbered 1 through N, and the n^{th} consumer's demand for good i is given by $x_i^n(p, w^n)$, where w^n is consumer n's initial wealth. In this case, total demand for good i can be written as:

$$\tilde{D}_i\left(p, w^1, ..., w^N\right) = \sum_{n=1}^{N} x_i^n(p, w^n).$$

However, notice that $\tilde{D}_i()$ gives total demand for good i as a function of prices and the wealth levels of the n consumers. As I said earlier, often we will not have access to information about individuals, only aggregates. Thus we may ask the question of when there exists a function $D_i(p, w)$, where $w = \sum_{n=1}^{N} w^n$ is aggregate wealth, that represents the same behavior as $\tilde{D}_i\left(p, w^1, ..., w^N\right)$. A second question is when, given that there exists an aggregate demand function $D_i(p, w)$, the behavior it characterizes is rational. We ask this question in two ways: First, when will the behavior resulting from $D_i(p, w)$ satisfy WARP? Second, when will it be as if $D_i(p, w)$ were generated by a "representative consumer" who is herself maximizing preferences? Finally, we will ask if there is a representative consumer, in what sense is the well-being of the representative consumer a measure of the well-being of society?

4.2.1 The Gorman Form

The major theme that runs through our discussion in this section is that in order for demand to aggregate, each individual's utility function must have an indirect utility function of the **Gorman**

Form. So, let me take a moment to introduce the terminology before we need it. An indirect utility function for consumer n, $v^n(p,w)$, is said to be of the **Gorman Form** if it can be written in terms of functions $a^n(p)$, which may depend on the specific consumer, and $b(p)$, which does not depend on the specific consumer:

$$v^n(p,w) = a^n(p) + b(p)\, w^n.$$

That is, an indirect utility function of the Gorman form can be separated into a term that depends on prices and the consumer's identity but not on her wealth, and a term that depends on a function of prices that is common to all consumers that is multiplied by that consumer's wealth.

The special nature of indirect utility functions of the Gorman Form is made apparent by applying Roy's identity:

$$x_i^n(p,w^n) = -\frac{\frac{\partial v^n}{\partial p_i}}{\frac{\partial v^n}{\partial w^n}} = -\frac{a_i^n(p) + \frac{\partial b(p)}{\partial p_i} w^n}{b(p)}. \tag{4.1}$$

From now on, we will let $b_i(p) = \frac{\partial b(p)}{\partial p_i}$. Now consider the derivative of a particular consumer's demand for commodity i : $\frac{\partial x_i^n(p,w^n)}{\partial w} = \frac{b_i(p)}{b(p)}$. This implies that wealth-expansion paths are given by:

$$\frac{\frac{\partial x_i^n(p,w^n)}{\partial w^n}}{\frac{\partial x_j^n(p,w^n)}{\partial w^n}} = \frac{b_i(p)}{b_j(p)}.$$

Two important properties follow from these derivatives. First, for a fixed price, p, $\frac{\partial x_i^n(p,w^n)}{\partial w}$ does not depend on wealth. Thus, as wealth increases, each consumer increases her consumption of the goods at a linear rate. The result is that each consumer's wealth-expansion paths are straight lines. Second, $\frac{\partial x_i^n(p,w^n)}{\partial w}$ is the same for all consumers, since $\frac{b_i(p)}{b(p)}$ does not depend on n. This implies that the wealth-expansion paths for different consumers are parallel (see MWG Figure 4.B.1).

Next, let's aggregate the demand functions of consumers with Gorman form indirect utility functions. Sum the individual demand functions from (4.1) across all n to get aggregate demand:

$$\begin{aligned} D_i\left(p, w^1, ..., w^n\right) &= \sum_n \frac{-a_i^n(p) - b_i(p)\, w^n}{b(p)} = \sum_n \frac{-a_i^n(p)}{b(p)} - \frac{b_i(p)}{b(p)} \sum w^n \\ &= \sum_n \frac{-a_i^n(p)}{b(p)} - \frac{b_i(p)}{b(p)} w^{total}. \end{aligned}$$

Thus if all consumers have utility functions of the Gorman form, demand can be written solely as a function of prices and total wealth. In fact, this is a necessary and sufficient condition: Demand can be written as a function of prices and total wealth if and only if all consumers have indirect utility functions of the Gorman form (see MWG Proposition 4.B.1).

As a final note on the Gorman form, recall the examples of quasilinear and homothetic utility we did earlier. It is straightforward to verify (at least in the examples) that if all consumers have identical homothetic preferences or if consumers have (not necessarily identical) preferences that are quasilinear with respect to the same good, then their preferences will be representable by utility functions of the Gorman form.

4.2.2 Aggregate Demand and Aggregate Wealth

I find the notation in the book in this section somewhat confusing. So, I will stick with the notation used above. Let $x_i^n(p, w^n)$ be the demand by consumer n for good i when prices are p and wealth is w^n, and let $\tilde{D}_i\left(p, w^1, ..., w^N\right)$ denote aggregate demand as a function of the entire vector of wealths.[2]

The general question we are asking here is whether or not the distribution of wealth among the consumers matters. If the distribution of wealth affects total demand for the various commodities, then we will be unable to write total demand as a function of prices and total wealth. On the other hand, if total demand does not depend on the distribution of wealth, we will be able to do so.

Let prices be given by $\bar{p}$ and the initial wealth for each consumer be given by $\bar{w}^n$. Let dw be a vector of wealth changes where dw^n represents the change in consumer $n's$ wealth and $\sum_{n=1}^N dw^n = 0$. Thus dw represents a redistribution of wealth among the n consumers. If total demand can be written as a function of total wealth and prices, then

$$\sum_{n=1}^{N} \frac{\partial x_i^n(p, \bar{w}^n)}{\partial w^n} dw^n = 0$$

for all i. If this is going to be true for all initial wealth distributions $\left(\bar{w}^1, ..., \bar{w}^N\right)$ and all possible rearrangements dw, it must be the case that partial derivative of demand with respect to wealth is equal for every consumer and every distribution of wealth:

$$\frac{\partial x_i^n(p, w^n)}{\partial w^n} = \frac{\partial x_i^m(p, w^m)}{\partial w^m}.$$

But, this condition is exactly the condition that at any price vector p, and for any initial distribution of wealth, the wealth effects of all consumers are the same. Obviously, if this is true then the changes

[2] One should be careful not to confuse the superscript with an exponent here. We are concerned with the question of when aggregate demand can be written as $D_i\left(p, \sum_{n=1}^N w^n\right)$, a function of prices and the total wealth of all consumers.

in demand as wealth is shifted from one consumer to another will cancel out. In other words, only total wealth (and not the distribution of wealth) will matter in determining total demand. And, this is equivalent to the requirement that for a fixed price each consumer's wealth expansion path is a straight line (since $\frac{\partial x_i^n(p,w^n)}{\partial w^n}$ and $\frac{\partial x_j^n(p,w^n)}{\partial w^n}$ must be independent of w^n) and that the slope of the straight line must be the same for all consumers (since $\frac{\partial x_i^n(p,w^n)}{\partial w^n} = \frac{\partial x_i^m(p,w^m)}{\partial w^m}$).

And, as shown in the previous section, this property holds if and only if consumers' indirect utility functions take the Gorman form. Hence if we allow wealth to take any possible initial distribution, aggregate demand depends solely on prices and total wealth if and only if consumers' indirect utility functions take the Gorman form.

To the extent that we prefer to look at aggregate demand or are unable to look at individual demand (because of data problems), the previous result is problematic. There are a whole lot of utility functions that don't take the Gorman form. There a number of approaches that can be taken:

1. We can weaken the requirement that aggregate demand depend only on total wealth. For example, if we allow aggregate demand to depend on the empirical distribution of wealth but not on the identity of the individuals who have the wealth, then demand can be aggregated whenever all consumers have the same utility function.

2. We required that aggregate demand be written as a function of prices and total wealth for any distribution of initial wealth. However, in reality we will be able to put limits on what the distributions of initial wealth look like. It may then be possible to write aggregate demand as a function of prices and aggregate wealth when we restrict the initial wealth distribution. One situation in which it will always be possible to write demand as a function of total wealth and prices is when there is a rule that tells you, given prices and total wealth, what the wealth of each individual should be. That is, if for every consumer n, there exists a function $w^n(p,w)$ that maps prices p and total wealth w to individual wealth w^n. Such a rule would exist if individual wealth were determined by government policies that depend only on p and w. We call this kind of function a wealth distribution rule.

    (a) An important implication of the previous point is that it always makes sense to think of aggregate demand when the vector of individual wealths is held fixed. Thus if we are only interested in the effects of price changes, it makes sense to think about their aggregate effects. (This is because $w^n(p,w) = \bar{w}^n$ for all p and w.)

4.2.3 Does individual WARP imply aggregate WARP?

The next aggregation question we consider is whether the fact that individuals make consistent choices implies that aggregate choices will be consistent as well. In terms of our discussion in Chapter 2, this involves the question of whether, when the Walrasian demand functions of the N consumers satisfy WARP, the resulting aggregate demand function will satisfy WARP as well. The answer to this question is, "Not necessarily."

To make things concrete, assume that there is a wealth distribution rule, so that it makes sense to talk about aggregate demand as $D(p,w) = (D_1(p,w), ..., D_L(p,w))$. In fact, to keep things simple, assume that the wealth distribution rule is that $w^n(p,w) = a_n w$. Thus consumer n is assigned a fraction a_n of total wealth. Thus

$$D(p,w) = \sum_n x^n(p,w^n).$$

The aggregate demand function satisfies WARP if, for any two combinations of prices and aggregate wealth, (p,w) and (p',w'), if $p \cdot D(p',w') \leq w$ and $D(p,w) \neq D(p',w')$, then $p' \cdot D(p,w) > w'$. This is the same definition of WARP as before.

The reason why individual WARP is not sufficient for aggregate WARP has to do with the Compensated Law of Demand (CLD). Recall that an individual's behavior satisfies WARP if and only if the CLD holds for all possible compensated price changes. The same is true for aggregate WARP. The aggregate will satisfy WARP if and only if the CLD holds in the aggregate for all possible compensated price changes. The problem is that just because a price change is compensated in the aggregate, it does not mean that the price change is compensated for each individual. Because of this, it does not necessarily follow from the fact that each individual's behavior satisfies the CLD that the aggregate will as well, since compensated changes in the aggregate need not imply compensated changes individually. See Example 4.C.1 and Figure 4.C.1 in MWG.

To make this a little more concrete without going into the details of the argument, think about how you would prove this statement: "If individuals satisfy WARP then the aggregate does as well." The steps would be:

1. Consider a compensated change in aggregate wealth.

2. This can be written as a sum of compensated changes in individual wealths.[3]

[3] Of course, this step is not true!

3. Individuals satisfy WARP if and only if they satisfy the CLD.

4. So, each individual change satisfies the CLD.

5. Adding over individual changes, the aggregate satisfies the CLD as well.

This proof is clearly flawed since step 2 is not valid. As shown above, it is not possible to write every price change that is compensated in the aggregate in terms of price changes that are compensated individual-by-individual. So, it turns out that satisfying WARP and therefore the CLD is not sufficient for aggregate decisions to satisfy WARP. However, if we impose stronger conditions on individual behavior, we can find a property that aggregates. That property is the **Uncompensated Law of Demand (ULD)**. The ULD is similar to the CLD, but it involves uncompensated changes. Thus a demand function $x(p,w)$ satisfies the ULD if for any price change $p \to p'$ the following holds:

$$\left(p' - p\right)\left(x\left(p', w\right) - x\left(p, w\right)\right) \leq 0.$$

Note the following:

1. If a consumer's demand satisfies the ULD, then it satisfies the CLD as well.

2. Unlike the CLD, the ULD aggregates. Thus if each consumer's demand satisfies the ULD, the aggregate demand function will as well.

Hence even though satisfaction of the CLD individually is not sufficient for the CLD in the aggregate, the ULD individually is sufficient for the ULD in the aggregate. So, the ULD individually implies WARP in the aggregate.

If we want to know which types of utility functions imply aggregate demand functions that satisfy WARP, we need to find those that satisfy the ULD. It turns out that homothetic preferences satisfy the ULD. Thus if each consumer has homothetic preferences, the implied aggregate demand will satisfy WARP.

In general, there is a calculus test to determine if a utility function satisfies the ULD property. It is given in MWG, and my advice is that if you ever need to know about such things, you look it up at that time. Basically, it has to do with making sure that wealth effects are not too strange (recall the example of the Giffen good – where the wealth effect leads to an upward sloping demand curve – the same sort of thing is a concern here).

4.2.4 Representative Consumers

The final question is when can the aggregate demand curve be used to make welfare measurements? In other words, when can we treat aggregate demand as if it is generated by a fictional "representative consumer," and when will changes in the welfare of that consumer correspond to changes in the welfare of society as a whole?

The first part of this question is, when is there a rational preference relation $\succeq$ such that the aggregate demand function corresponds to the Walrasian demand function generated by these preferences? If such a preference relation exists, we say that there is a **positive representative consumer.**

The first necessary condition for the existence of a positive representative consumer is that it makes sense to aggregate demand. Thus consumers must have indirect utility functions of the Gorman form (or wealth must be assigned by a wealth-assignment rule). In addition, the demand must correspond to that implied by the maximization of some rational preference relation. In essence, we need the Slutsky matrix to be negative semi-definite as well.

An additional question is whether the preferences of the positive representative consumer capture the welfare of society as a whole. This is the question of whether the positive representative consumer is **normative** as well. For example, suppose there is a **social welfare function** $W(u_1, ..., u_N)$ that maps utility levels for the N consumers to real numbers and such that utility vectors assigned higher numbers are better for the society than vectors assigned lower numbers. Thus $W()$ is like a utility function for the society. Now suppose that for any level of aggregate wealth we assign wealth to the consumers in order to maximize W. That is, $w^1, ..., w^N$ solves

$$\max_{w^1,...,w^N} W\left(v^1\left(p, w^1\right), ..., v^N\left(p, w^N\right)\right)$$

$$s.t. \sum_{n=1}^{N} w^n \leq w.$$

Thus it corresponds to the situation where a benevolent dictator distributes wealth in the society in order to maximize social welfare. This defines a wealth assignment rule, so we know that aggregate demand can be represented as a function of p and total wealth w.

In the case where wealth is assigned as above, not only can demand be written as $D(p, w)$, but also these demand functions are consistent with the existence of a positive representative consumer. Further, if the aggregate demand functions are generated by solving the previous program, they have welfare significance and can be used to make welfare judgments (using the techniques from

Chapter 3).

An important social welfare function is the **utilitarian social welfare function.** The utilitarian social welfare function says that social welfare is the sum of the utilities of the individual consumers in the economy. Now, assume that all consumers have indirect utility functions of the Gorman Form: $v^n(p, w^n) = a^n(p) + b(p)w^n$. Using the utilitarian social welfare function implies that the social welfare maximization problem is:

$$\begin{aligned} &\max \sum v^n(p, w^n) \\ s.t. \quad &: \sum w^n \leq w. \end{aligned}$$

But, this can be rewritten as:

$$\begin{aligned} &\max \left(\sum a^n(p)\right) + b(p) \sum w^n \\ s.t. \quad &: \sum w^n \leq w, \end{aligned}$$

and any wealth assignment rule that fully distributes wealth, $\sum w^n(p, w) = w$, solves this problem. The result is this: When consumers have indirect utility of the Gorman Form (with the same $b(p)$), aggregate demand can always be thought of as being generated by a normative representative consumer with indirect utility function $v(p, w) = \sum_n a^n(p) + b(p)w$, who represents the utilitarian social welfare function.

In fact, it can be shown that when consumers' preferences have Gorman Form indirect utility functions, then $v(p, w) = \sum_n a^n(p) + b(p)w$ is an indirect utility function for a normative representative consumer **regardless of the form of the social welfare function**.[4] In addition, when consumers have Gorman Form utility functions, the indirect utility function is also independent of the particular wealth distribution rule that is chosen.[5]

This is all I want to say on the subject for now. The main takeaway message is that you should be careful about dealing with aggregates. Sometimes they make sense, sometimes they do not. And, just because they make sense in one way (i.e., you can write demand as $D(p, w)$), they may not make sense in another (i.e., there is a positive or normative consumer).

[4] This not generally true when consumers' preferences are not Gorman-form. The preferences of the normative representative consumer will depend on the particular social welfare function used to generate those preferences.

[5] Again, this property will not hold if consumers' preferences cannot be represented by a Gorman form utility function.

4.3 The Composite Commodity Theorem

There are many commodities in the world, but usually economists will only be interested in a few of them at any particular time. For example, if we are interested in studying the wheat market, we may divide the set of commodities into "wheat" and "everything else." In a more realistic setting, an empirical economist may be interested in the demand for broad categories of goods such as "food," "clothing," "shelter," and "everything else." In this section, we consider the question of when it is valid to group commodities in this way.[6]

To make things simple, consider a three-commodity model. Commodity 1 is the commodity we are interested in, and commodities 2 and 3 are "everything else." Denote the initial prices of goods 2 and 3 by p_2^0 and p_3^0, and suppose that if prices change, the relative price of goods 2 and 3 remain fixed. That is, the price of goods 2 and 3 can always be written as $p_2 = tp_2^0$ and $p_3 = tp_3^0$, for $t \geq 0$. For example, if good 2 and good 3 are apples and oranges, this says that whenever the price of apples rises, the price of oranges also rises by the same proportion. Clearly, this assumption will be reasonable in some cases and unreasonable in others, but for the moment will will assume that this is the case.

The consumer's expenditure minimization problem can be written as:

$$\begin{aligned} & \min_{x \geq 0} p_1 x_1 + tp_2^0 x_2 + tp_3^0 x_3 \\ s.t. \quad : \quad & u(x) \geq u. \end{aligned}$$

Solving this problem yields Hicksian demand functions $h\left(p_1, tp_2^0, tp_3^0, u\right)$ and expenditure function $e\left(p_1, tp_2^0, tp_3^0, u\right)$.

Now, suppose that we are interested only in knowing how consumption of good 1 depends on t. In this case, we can make the following change of variables. Let $y = p_2^0 x_2 + p_3^0 x_3$. Thus y is equal to expenditure on goods 2 and 3, and t then corresponds to the "price" of this expenditure. As t increases, y becomes more expensive. Applying this change of variable to the $h()$ and $e()$ yields the new functions:

$$\begin{aligned} h^*(p_1, t, u) &\equiv h\left(p_1, tp_2^0, tp_3^0, u\right) \\ e^*(p_1, t, u) &\equiv e\left(p_1, tp_2^0, tp_3^0, u\right). \end{aligned}$$

[6] References: Silberberg, Section 11.3; Deaton and Muellbauer *Economics and Consumer Behavior,* pp. 120-122; Jehle and Reny, p. 266.

It remains to be shown that $h^*()$ and $e^*()$ satisfy the properties of well-defined compensated demand and expenditure functions (see Section 3.4). For $e^*(p_1, t, u)$, these include:

1. Homogeneity of degree 1 in p
2. Concavity in (p_1, t) (i.e. the Slutsky matrix is negative semi-definite)
3. $\frac{\partial e^*}{\partial t} = y$ (and the other associated derivative properties)

In fact, these relationships can be demonstrated. Hence we have the **composite commodity theorem**:

Theorem 8 *When the prices of a group of commodities move in parallel, then the total expenditure on the corresponding group of commodities can be treated as a single good.*

The composite commodity theorem has a number of important applications. First, the composite commodity theorem can be used to justify the two-commodity approach that is frequently used in economic models. If we are interested in the effect of a change in the price of wheat on the wheat market, assuming that all other prices remain fixed, the composite commodity theorem justifies treating the world as consisting of wheat and the composite commodity "everything else."

A second application of the composite commodity theorem is to models of consumption over time, which we will cover later (see Section 4.6 of these notes). Since the prices of goods in future periods will tend to move together, application of the composite commodity theorem allows us to analyze consumption over time in terms of the composite commodities "consumption today," "consumption tomorrow," etc.

4.4 So Were They Just Lying to Me When I Studied Intermediate Micro?

Recall from your intermediate microeconomics course that you probably did welfare evaluation by looking at changes in Marshallian consumer surplus, the area to the left of the aggregate demand curve. But, I've told you that: a) consumer surplus is not a good measure of the welfare of an individual consumer; b) even if it were, it usually doesn't make sense to think of aggregate demand as depending only on aggregate wealth (which it does in the standard intermediate micro model); and c) even if it did, looking at the equivalent variation (which is better than looking at the change in consumer surplus) for the aggregate demand curve may not have welfare significance. So, at this point, most students are a little concerned that everything they learned in intermediate micro was wrong. The point of this interlude is to argue that this is not true. Although many of the assumptions made in order to simplify the presentation in intermediate micro are not explicitly stated, they can be explicitly stated and are actually pretty reasonable.

To begin, note that the point of intermediate micro is usually to understand the impact of changes on one or a few markets. For example, think about the change in the price of apples on the demand for bananas. It is widely believed that since expenditure on a particular commodity (like apples or bananas) is usually only a small portion of a consumer's budget, the income effects of changes in the prices of these commodities are likely to be small. In addition, since we are looking at only a few price changes and either holding all other prices constant or varying them in tandem, we can apply the composite commodity theorem and think of the consumer's problem as depending on the commodity in question and the composite commodity "everything else." Thus the consumer can be thought of as having preferences over apples, bananas, and everything else.

Now, since the income effects for apples and bananas are likely to be small, a reasonable way to represent the consumer's preferences is as being quasilinear in "everything else." That is, utility looks like:

$$u(a, b, e) = f(a, b) + e$$

where $a = apples$, $b = bananas$, and $e = everything\ else$. Once we agree that this is a reasonable representation of preferences for our purposes, we can point out the following:

1. Since there are no wealth effects for apples or bananas, the Walrasian and Hicksian demand curves coincide, and the change in Marshallian consumer surplus is the same as EV. Hence ΔCS is a perfectly fine measure of changes in welfare.

2. If all individual consumers in the market have utility functions that are quasilinear in everything else, then it makes sense to write demand as a function of aggregate wealth, since quasilinear preferences can be represented by indirect utility functions of the Gorman form.

3. Since all individuals have Gorman form indirect utility functions, then aggregate demand can always be thought of as corresponding to a representative consumer for a social welfare function that is utilitarian. Thus ΔCS computed using the aggregate demand curve has welfare significance.

Thus, by application of the composite commodity theorem and quasilinear preferences, we can save the intermediate micro approach. Of course, our ability to do this depends on looking at only a few markets at a time. If we are interested in evaluating changes in many or all prices, this may not be reasonable. As you will see later, this merely explains why partial equilibrium is a topic for intermediate micro and general equilibrium is a topic for advanced micro.

4.5 Consumption With Endowments

Until now we have been concerned with consumers who are endowed with initial wealth w. However, an alternative approach would be to think of consumers as being endowed with both wealth w and a vector of commodities $a = (a_1, ..., a_L)$, where a_i gives the consumer's initial endowment of commodity i.[7] In this case, the consumer's UMP can be written as:

$$\begin{aligned} &\max_x u(x) \\ s.t. \quad : \quad & p \cdot x \leq p \cdot a + w \end{aligned}$$

The value of the consumer's initial assets is given by the sum of her wealth and the value of her endowment, $p \cdot a$. Thus the mathematical approach is equivalent to the situation where the consumer first sells her endowment and then buys the best commodity bundle she can afford at those prices.

The first-order conditions for this problem are found in the usual way. The Lagrangian is given by:

$$L = u(x) + \lambda (p \cdot a + w - p \cdot x)$$

implying optimality conditions:

$$\begin{aligned} u_i - \lambda p_i &= 0 : i = 1, ..., L. \\ p \cdot x - p \cdot a - w &\leq 0. \end{aligned}$$

Denote the solution to this problem as

$$x(p, w, a)$$

where w is non-endowment wealth and a is the consumer's initial endowment.

[7] Reference: Silberberg (3rd edition), pp. 299-304.

We can also solve a version of the expenditure minimization problem in this context. Consider the problem:

$$\min_x p \cdot x - p \cdot a$$
$$s.t. \quad : \quad u(x) \geq u.$$

The objective function in this model is non-endowment wealth. Thus it plays the role of w in the UMP, and the question asked by this problem can be stated as: How much non-endowment wealth is needed to achieve utility level u when prices are p and the consumer is endowed with a?

The endowment a drops out of the Lagrangian when you differentiate with respect to x_i. Hence the non-endowment expenditure minimizing bundle (NEEMB) is not a function of a. We'll continue to denote it as $h(p, u)$. However, while the NEEMB does not depend on a, the non-endowment expenditure function does. Let

$$e^*(p, u, a) \equiv p \cdot (h(p, u) - a).$$

Again, $e^*(p, u, a)$ represents the non-endowment wealth necessary to achieve utility level u as a function of p and endowment a. By the envelope theorem (or the derivation for $\frac{\partial e}{\partial p_i} = h_i(p, u)$ we did in Section 3.4.3) it follows that

$$\frac{\partial e^*}{\partial p_i} \equiv h_i(p, u) - a_i.$$

Thus the sign of $\frac{\partial e^*}{\partial p_i}$ depends on whether $h_i(p, u) > a_i$ or $h_i(p, u) < a_i$. If $h_i(p, u) > a_i$ the consumer is a net purchaser of good i, consuming more of it than her initial endowment. If this is the case, then an increase in p_i increases the cost of purchasing the good i from the market, and this increases total expenditure at a rate of $h_i(p, u) - a_i$. On the other hand, if $h_i(p, u) < a_i$, then the consumer is a net seller of good i, consuming less of it than her initial endowment. In this case, increasing p_i increases the revenue the consumer earns by selling the good to the market. The result is that the non-endowment wealth the consumer needs to achieve utility level u decreases at a rate of $|h_i(p, u) - a_i|$.

Now, let's rederive the Slutsky equation in this environment. The following identity relates $h()$ and $x()$:

$$h_i(p, u) \equiv x_i(p, e^*(p, u, a), a).$$

Differentiating with respect to p_j yields:

$$\begin{aligned}\frac{\partial h_i}{\partial p_j} &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\frac{\partial e^*}{\partial p_j} \\ &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\left(h_j\left(p,u\right) - a_j\right) \\ &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\left(x_j\left(p,w,a\right) - a_j\right).\end{aligned}$$

A useful reformulation of this equation is:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial w}\left(x_j\left(p,w,a\right) - a_j\right).$$

The difference between this version of the Slutsky equation and the standard form is in the wealth effect. Here, the wealth effect is weighted by the consumer's net purchase of good i.[8] So, think about a consumer who is endowed with a_1 units of good 1 and faces an increase in p_1. For concreteness, say that good 1 is gold, I am the consumer, and we are interested in my purchases of new ties (good 2) in response to a change in the price of gold. If the price of gold goes up, I will tend to purchase more ties if we assume that ties and gold are substitutes in my utility function. This means that $\frac{\partial h_2}{\partial p_1} > 0$. However, an increase in the price of gold will also have a wealth effect. Whether this effect is positive or negative depends on whether I am a net purchaser or net seller of gold. If I buy more gold than I sell, then the price increase will be bad for me. In terms of the Slutsky equation, this means $(x_1 - a_1) > 0$. For a normal good ($\frac{\partial x_2}{\partial w} > 0$), this means that $\frac{\partial x_2}{\partial p_1}$ will be smaller than $\frac{\partial h_2}{\partial p_1}$ – I shift consumption towards ties due to the price change, but the price increase in gold makes me poorer so I don't increase tie consumption quite as much as in a compensated price change.

If I am a net seller of gold, an increase in the price of gold has a positive effect on my wealth. Since I am selling gold to the market, increasing its price p_1 actually makes me wealthier in proportion to $(a_1 - x_1)$. And, since the price change makes me wealthier (because $x_1 - a_1 < 0$), the effect of the whole wealth/endowment term is subtracting a negative number (again, assuming ties are a normal good). Thus $\frac{\partial x_2}{\partial p_1}$ will be greater than $\frac{\partial h_2}{\partial p_1}$, and I will consume more ties due to both the price substitution effect and the fact that the price change makes me wealthier.

[8] Actually, there is no difference between this relationship and the standard Slutsky equation. The standard model is equivalent to this model where $a = (0, ..., 0)$. If you insert these values into the Slutsky equation with endowments, you get the exactly the standard version of the Slutsky equation.

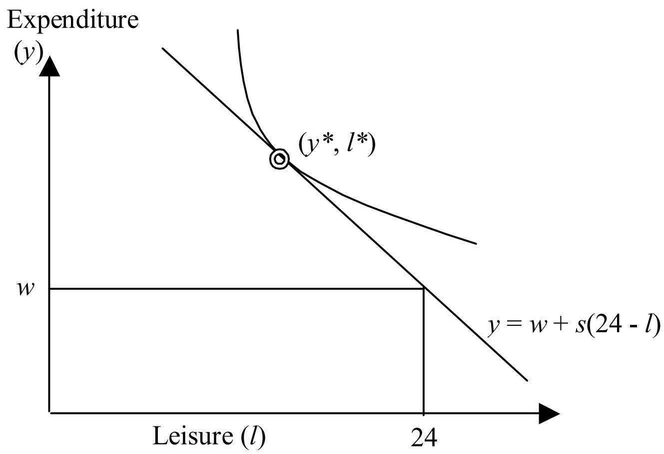

Figure 4.3: Labor-Leisure Choice

Thus the main difference between the standard model and the endowment model lies in this adjustment to the Slutsky matrix: The wealth effect must be adjusted by whether a consumer is a net purchaser or a net seller of the good in question. This has important applications in general equilibrium theory (which we'll return to much later), as well as applications in applied consumption models. We turn to one such example here.

4.5.1 The Labor-Leisure Choice

As an application of the previous section, consider a consumer's choice between labor and leisure. We are interested in the consumer's leisure decision, so we'll apply the composite commodity theorem and model the consumer as caring about leisure, l, and everything else, y. Let the consumer's utility function be

$$u(y, l).$$

If the wage rate is s, w is non-endowment wealth, and the price of "everything else" is normalized to 1, the consumer's budget constraint is given by:

$$y \leq s(24 - l) + w.$$

The solution to this problem is given by the point of tangency between the utility function and the budget set. This point is illustrated in the Figure 4.3.

Initially, the consumer is endowed with 24 hours of leisure per day. Since the consumer cannot consume more than 24 hours of leisure per day, at the optimum the consumer must be a net seller

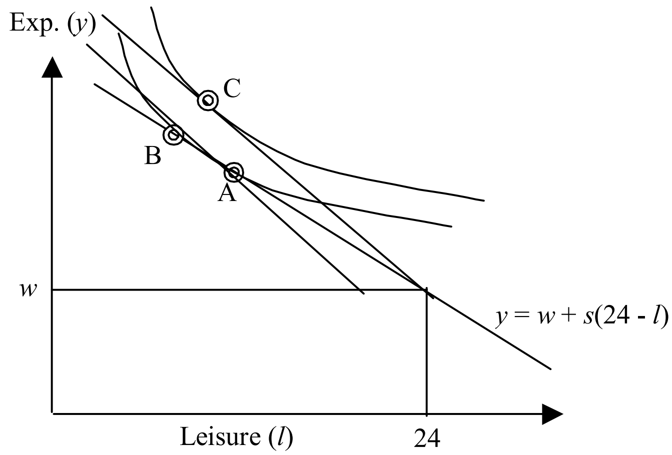

Figure 4.4: A Wage Increase

of leisure. Thus an increase in the price of leisure, s, increases the consumer's wealth. Hence the compensation must be negative. A compensated increase in the price of leisure is illustrate in Figure 4.4. At the original wage rate the consumer maximizes utility by choosing the bundle at point A. Since the consumer is a net seller of leisure, the compensated change in demand for leisure is negative. So, when compensated for the price change, the consumer's choice moves from point A to point B, and she consumes less leisure at the higher wage rate. However, since the consumer is a net seller of leisure, the compensation is negative. Hence when going from the compensated change to uncompensated change we move from point B to point C. That is, the wealth effect here leads to the consumer consuming more leisure than before the compensation took place.

Let's think of this another way. Suppose that wages increase. Since you get paid more for every additional hour you work, you will tend to work more (which means that you will consume less leisure). However, since you make more for every hour you work, you also get paid more for all of the hours you are already working. This makes you wealthier, and because of it you will tend to want to work less (that is, consume more leisure, assuming it is a normal good). Thus the income effect and substitution effect work in opposite directions here precisely because the consumer is a net seller of leisure. This is in contrast with consumer theory without endowments, where you decrease consumption of a normal good whose price has increased, both because its relative price has increased and because this increase has made you poorer.

Note that it is also possible to get a Giffen-good like phenomenon here even though leisure is a normal good. This happens if the income effect is much larger than the substitution effect,

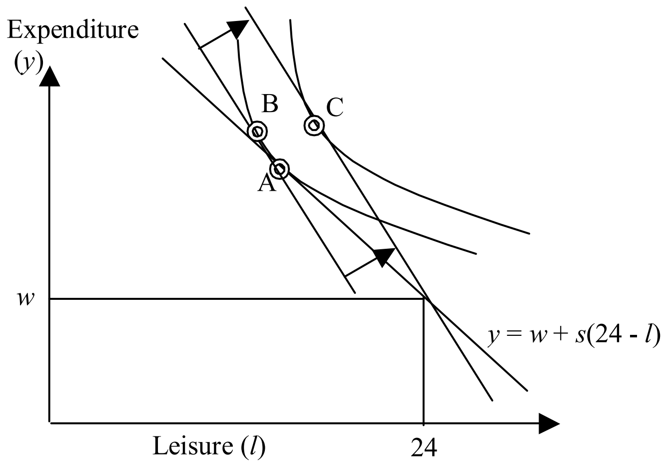

Figure 4.5: Positive Labor Supply Elasticity

as in Figure 4.5, where the arrows depict the large income effect (point B to point C). As an illustration, think of the situation in which a person earns minimum wage, let's say \$5 per hour, and chooses to work 60 hours per week. That gives total wages of \$300 per week. If the government raises the minimum wage by \$1 per hour, this increases the consumer's total wages to \$360, a 20% increase. The consumer likely has two responses to this. Since the consumer gets paid more for each additional hour of work, she may decide to work more hours (since she will be willing to give up more leisure at the higher wage rate). However, since the \$1 increase in wages has increased total wage revenue by 20% already, this may make the consumer work less, since she is already richer than before. In situations where the change in total wages is large relative to the wage rate (i.e., the consumer is working a lot of hours), the latter effect may swamp the former.

There have been many studies of this labor-leisure tradeoff in the U.S. They are frequently associated with worries over whether raising taxes on the wealthy will cause them to cut their labor supply. My understanding of the evidence (through conversations with labor economists mostly) is that labor supply elasticities are positive but small, similar to the depiction in Figure 4.5.[9]

[9] In fact, labor supply elasticities tend to be pretty small for men, larger for women, but always positive (i.e. an increase in wages - or a cut in income taxes - leads people to work more).

4.6 Consumption Over Time

Up until now we have been considering a model of consumption that is static. Time does not enter into our model at all. This model is very useful for modeling a consumer's behavior at a particular point in time. It is also useful for modeling the consumer's behavior in two different situations. This is what we called "comparative statics." However, as the name suggests, even though the consumer's behavior in two different situations can be compared using the static model, we are really just comparing two static situations: No attempt is made to model how the consumer's behavior evolves over time.

While the static model is useful for answering some questions, often we will be interested specifically in the consumer's consumption decisions over time. For example, will the consumer borrow or save? Will her consumption increase or decrease over time? How are these conclusions affected by changes in exogenous parameters such as prices, interest rates, or wealth?

Fortunately, we can adapt our model of static consumption to consider dynamic situations. There are two key features of the dynamic model that need to be addressed. First, the consumer may receive her wealth over the course of her lifetime. But, units of wealth today and units of wealth tomorrow are not worth the same to the consumer. Thus we must come up with a way to measure wealth received (or spent) at different times. Second, there are many different commodities sold and consumed during each time period. Explicitly modeling every commodity would be difficult, and it would make it harder to evaluate broad trends in the consumer's behavior, which is what we are ultimately interested in.

The solution to these problems is found in the applications of consumer theory that we have been developing. The first step is to apply the composite commodity theorem. Since prices at a particular time tend to move in unison, we can combine all goods bought at a particular time into a composite commodity, "consumption at time t." We can then analyze the dynamic problem as a static problem in which the commodities are "consumption today," "consumption tomorrow," etc. The problem of wealth being received over time is addressed by adding endowments to the static model. Thus the consumer's income (addition to wealth) during period t can be thought of as the consumer's endowment of the composite commodity "consumption at time t." The final issue, that of capturing the fact that a unit of wealth today is worth more than a unit of wealth tomorrow, is addressed by assigning the proper prices to consumption in each period. This is done through a process known as **discounting.**

4.6.1 Discounting and Present Value

Suppose that you have \$1 today that you can put in the bank. The interest rate the bank pays is 10% per year. If you invest this dollar, you have \$1.10 at the end of the year. On the other hand, suppose that you need to have \$1 at the end of the year. How much should you invest today in order to make sure that you have \$1 at the end of the year? The answer to this question is given by the solution to the equation:

$$\begin{aligned}(1+.1)\,y &= 1\\ y &= \frac{1}{1+.1} \simeq 0.91.\end{aligned}$$

Thus in order to make sure you have \$1 a year from now, you should invest 91 cents today.

To put the question of the previous paragraph another way, if I were to offer you \$1 a year from now or y dollars today, how large would y have to be so that you are just indifferent between the dollar in a year and y today? The answer is $y = 0.91$ (assuming the interest rate is still 10%).[10] Thus we call \$0.91 the **present value** of \$1 a year from now because it is the value, in current dollars, of the promise of \$1 in a year.

In fact, we can think of the 91 cents in another way. We can also think of it as the price, in current dollars, of \$1 worth of consumption a year from now. In other words, if I were to offer to buy you \$1 worth of stuff a year from now and I wanted to break even, I should charge you a price of 91 cents.

The concept of present value can also be used to convert streams of wealth received over multiple years into their current-consumption equivalents. Suppose we call the current period 0, and that the world lasts until period T. If the consumer receives a_t dollars in period t, and the interest rate is r (and remains constant over time), then the present value of this stream of payments is given by:

$$PV_a = a_0 + \sum_{t=1}^{T} \frac{a_t}{(1+r)^t} = \sum_{t=0}^{T} \delta^t a_t. \tag{4.2}$$

where $\delta = \frac{1}{1+r}$ is the **discount factor**.[11] But, this can also be thought of as a problem of consumption with endowments. Let the commodities be denoted by $x = (x_0, ..., x_T)$, where x_t is consumption in period t (by application of the composite commodity theorem). Let a_t be the consumer's endowment of the consumption good in period t. Then, if we let the price of

[10] This answer ignores the issue of impatience, which we will address shortly.

[11] In the event that the interest rate changes over time, the interest rate r can be replaced with the period-specific interest rate, r_t, and the discount rate is then $\delta_t = \frac{1}{1+r_t}$.

consumption in period t, denoted p_t, be $p_t = \frac{1}{(1+r)^t}$, the present value formula above can be written as:

$$PV_a = \sum_{t=0}^{T} p_t a_t = p \cdot a$$

where $p = (p_0, ..., p_T)$ and $a = (a_0, ..., a_T)$. But, this is exactly the expression we had for endowment wealth in the model of consumption with endowments. This provides the critical link between the static model and the dynamic model.

4.6.2 The Two-Period Model

We now show how the approach developed in the previous section can be used to develop a model of consumption over time. Suppose that the consumer lives for two periods: today (called period 0) and tomorrow (called period 1). Let x_0 and x_1 be consumption in periods $t = 0$ and $t = 1$, respectively, and let a_0 and a_1 be income (endowment) in each period, measured in units of consumption. Suppose that the consumer can borrow or save at an interest rate of $r \geq 0$. Thus the price of consumption in period t (in terms of consumption in period 0) is given by $p_t = \frac{1}{(1+r)^t}$.

Assume that the consumer has preferences over consumption today and consumption tomorrow represented by utility function $u(x_0, x_1)$, and that this utility function satisfies all of the nice properties: It is strictly quasiconcave and strictly increasing in each of its arguments, and twice differentiable in each argument. The consumer's UMP can then be written as:

$$\begin{aligned} & \max_{x_0, x_1} u(x_0, x_1) \\ s.t & : \\ x_0 + \frac{x_1}{1+r} & \leq a_0 + \frac{a_1}{1+r} \end{aligned}$$

where, of course, the constraint is just another way of writing $p \cdot x \leq p \cdot a$, which just says that the present value of consumption must be less than the present value of the consumer's endowment. It is simply a dynamic version of the budget constraint.[12] Note that since $p_0 = 1$, the exogenous parameters in this problem are r and a. It is convenient to write them as $p_1 = \frac{1}{1+r}$ and a, however, and we will do this.

[12] Note that we could use the same model where the price of consumption in period t is not necessarily given by $p_t = \frac{1}{(1+r)^t}$. This approach will work whenever the price of consumption in period t in terms of consumption today is well-defined, even if it is not given by the above formula. The advantage of the discount-rate formulation is that it allows us to consider the impact of changes in the interest rate on consumption.

This problem can be solved using the standard Lagrangian methodology:

$$L = u(x_0, x_1) + \lambda \left(a_0 + \frac{a_1}{1+r} - x_0 - \frac{x_1}{1+r} \right).$$

Assuming an interior solution, first-order conditions are given by:

$$\begin{aligned} u_t &= \frac{\lambda}{(1+r)^t} : t \in \{0,1\}. \\ x_0 + \frac{x_1}{1+r} &\leq a_0 + \frac{a_1}{1+r}. \end{aligned}$$

Of course, as before, we know that the budget constraint will bind. This gives us our three equations in three unknowns, which can then be solved for the demand functions $x_t(p_1, a)$. The arguments of the demand functions are the exogenous parameters – interest rate r and endowment vector $a = (a_0, a_1)$. See Figure 12.1 in Silberberg for a graphical illustration – it's just the same as the standard consumer model, though.

We can also consider the expenditure minimization problem for the dynamic model. Earlier, we minimized the amount of non-endowment wealth needed to achieve a specified utility level. We do the same here, where non-endowment wealth is taken to be wealth in period 0.

$$\begin{aligned} \min a_0 &= x_0 + p_1(x_1 - a_1) \\ s.t. &: u(x) \geq u. \end{aligned}$$

The Lagrangian is given by:

$$L = x_0 + p_1(x_1 - a_1) - \lambda(u(x) - u).$$

The first-order conditions are derived as in the standard EMP, and the solution can be denoted by $h_t(r, u)$.[13] Let $a_0(p_1, a) = h_0(p_1, u) + p_1(h_1(p_1, u) - a_1)$ be the minimum wealth needed in period 0 to achieve utility level u when the interest rate is r.

Finally, we can link the solutions to the UMP and EMP in this context using the identity:

$$h_t(p_1, u) = x_t(p_1, a_0(p_1, u), a_1).$$

[13] Note that the endowment in period 1 (a_1) drops out when you differentiate. Hence the expenditure minimizing consumption bundle does not depend on a_1 (although the amount of period 0 wealth needed to purchase that consumption bundle will depend on a_1). Thus, h_t is not a function of a_1, but e is.

Differentiating with respect to p

$$\begin{aligned}\frac{\partial h_t}{\partial p_1} &= \frac{\partial x_t}{\partial p_1} + \frac{\partial x_t}{\partial a_0}\frac{\partial a_0}{\partial p_1} \\ &= \frac{\partial x_t}{\partial p_1} + \frac{\partial x_t}{\partial a_0}\left(h_1\left(p_1, u\right) - a_1\right) \\ &= \frac{\partial x_t}{\partial p_1} + \frac{\partial x_t}{\partial a_0}\left(x_1\left(p_1, a\right) - a_1\right).\end{aligned}$$

Using this version of the Slutsky equation, we can determine the effect of a change in the interest rate in each period. Let $t = 1$, and rewrite the Slutsky equation as:

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial h_1}{\partial p_1} + \frac{\partial x_1}{\partial a_0}\left(a_1 - x_1\right).$$

If r decreases, the price of future consumption (p_1) increases. We know that the compensated change in demand for future consumption $\frac{\partial h_1}{\partial p_1} \leq 0$. In fact, it is most likely negative: $\frac{\partial h_1}{\partial p_1} < 0$. The wealth effect depends on whether x_1 is normal or inferior (i.e., the sign of $\frac{\partial x_1}{\partial a_0}$) and whether the consumer saves in period 0 (implying $a_1 < x_1$) or borrows in period 0 (meaning $a_1 > x_1$). Since x_1 is all consumption in period 1, it only makes sense to think of it as normal. Hence if the consumer saves in period 0, the wealth effect will reinforce the compensated change in demand. Increasing the price of consumption in period 2 makes the consumer poorer (since saving the same amount yields less second period consumption than it did before the increase in p), and this will also lead her to decrease her consumption in period 1. Conversely, if the consumer borrows in period 0, then the increase in p makes the consumer wealthier since less consumption must be forfeited in period 1 to finance the same consumption in period 0. In this case, the effect on second period consumption is ambiguous: $\frac{\partial h_1}{\partial p_1}$ is negative, but the wealth effect is positive.

4.6.3 The Many-Period Model and Time Preference

This section has three aims: 1) To extend the two-period model of the previous section to a many-period model; 2) To incorporate into our model the idea that people's attitudes toward intertemporal substitution remain constant over time - we call this idea **dynamic consistency**; 3) To incorporate into our model the idea that people are impatient.

Extending the model to multiple periods is straightforward. Define utility over consumption in periods 0 through T as $U(x_0, ..., x_T)$. The UMP is then given by:[14]

$$\max_{x_0,...,x_T} U(x_0, ..., x_T)$$
$$s.t \ : \ \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \leq \sum_{t=0}^{T} \frac{a_t}{(1+r)^t}.$$

What does it mean for consumers to have dynamically consistent preferences, i.e., attitudes toward intertemporal substitution that remain constant over time? The idea is that your willingness to sacrifice a unit of consumption in period t_0 for a unit of consumption in period t_1 should depend only on the amount you are currently consuming in periods t_0 and t_1 and the amount of time between t_0 and $t_1 : t_1 - t_0$. For example, suppose the time period of consumption is 5 years, and that the consumer's current consumption path (which is not necessarily optimal) is given by:

	x_0	x_1	x_2	x_3	x_4
Consumption	10	20	5	10	20

If the consumer's attitudes toward intertemporal substitution remain constant, then the amount of consumption the consumer would be willing to give up in period 0 for an additional unit of consumption in period 1 should be the same as the amount of consumption the consumer is willing to give up in period 3 for an additional unit of consumption in period 4. This amount depends on the consumption in the two periods under consideration, 10 and 20 in each case, and on the amount of time between the periods, 1 in each case. Thus, for example, dynamic consistency implies that the consumer will prefer $x_0 = 11$, $x_1 = 19$, $x_2 = 5$, $x_3 = 10$, $x_4 = 20$ to the current consumption path if and only if she prefers $x_0 = 10$, $x_1 = 20$, $x_2 = 5$, $x_3 = 11$, and $x4 = 19$ to the current consumption path.

What we mean by impatience is this: Suppose I were to give you the choice between your favorite dinner today or the same dinner a year from now. Intuition about people as well as lots of experimental evidence tell us that almost everybody would rather have the meal today. Thus the meaning of impatience is that, all else being equal, consumers would rather consume sooner than later. Put another way, assume that you currently plan to consume the same amount today and tomorrow. The utility associated with an additional unit of consumption today is greater than the utility of an additional unit of consumption tomorrow.

[14] Note that utility over consumption paths here has been written as capital $U(x_0, ..., x_T)$. There will be a function called small $u()$ in a minute.

Impatience and dynamic consistency of preferences are most easily incorporated into our consumer model by assuming that the consumer's utility function can be written as:

$$U\left(x_0, ..., x_T\right) = \sum_{t=0}^{T} \frac{u\left(x_t\right)}{\left(1+\rho\right)^t},$$

where $u\left(x_t\right)$ gives the consumer's utility from consuming x_t units of output in period t and $\rho > 0$ is the consumer's **rate of time preference.** Note that lower-case $u\left(x\right)$ gives utility of consuming x_t in a single period, while capital $U\left(x_0, ..., x_T\right)$ is the utility from consuming consumption vector $\left(x_0, ..., x_T\right)$.

We can confirm that this utility function exhibits impatience and dynamic consistency in a straightforward manner. Impatience is easy. Consider two periods t_0 and t_1 such that $t_1 > t_0$ and $x_{t_0} = x_{t_1} = x^*$. Marginal utility in periods t_0 and t_1 are given by:

$$\begin{aligned} U_{t_0} &= \frac{u'\left(x^*\right)}{\left(1+\rho\right)^{t_0}} \\ U_{t_1} &= \frac{u'\left(x^*\right)}{\left(1+\rho\right)^{t_1}}. \end{aligned}$$

And, $U_{t_0} - U_{t_1} = u'\left(x^*\right)\left(\frac{1}{(1+\rho)^{t_0}} - \frac{1}{(1+\rho)^{t_1}}\right)$, which is positive whenever $t_1 > t_0$. Thus the consumer is impatient.

To check dynamic consistency, compute the consumer's marginal rate of substitution between two periods, t_0 and t_1 :

$$\frac{U_{t_1}}{U_{t_0}} = \frac{\frac{u'\left(x_{t_1}\right)}{(1+\rho)^{t_1}}}{\frac{u'\left(x_{t_0}\right)}{(1+\rho)^{t_0}}} = \frac{u'\left(x_{t_1}\right)}{u'\left(x_{t_0}\right)}\left(1+\rho\right)^{t_0 - t_1}.$$

Since the marginal rate of substitution depends only on the consumption in each period $x_{t_1}$and x_{t_0} and the amount of time between the two periods, $t_0 - t_1$, but not on the periods themselves, this utility function is also dynamically consistent.

Because it satisfies these two properties, we will use a utility function of the form:

$$U\left(x_0, ..., x_T\right) = \sum_{t=0}^{T} \frac{u\left(x_t\right)}{\left(1+\rho\right)^t},$$

for most of our discussion. We will assume that $U\left(x_0, ..., x_T\right)$ is strictly quasiconcave, and increasing and differentiable in each of its arguments.

Question: Does this mean that $u\left(\right)$ is concave? Answer: No!

In the multi-period version of the dynamic consumer model, the UMP can be written as:

$$\max_{x_0,\ldots,x_T} \sum_{t=0}^{T} \frac{u(x_t)}{(1+\rho)^t}$$
$$s.t. \quad : \quad \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \leq \sum_{t=0}^{T} \frac{a_t}{(1+r)^t}.$$

The Lagrangian is set up in the usual way, and the first-order conditions for an interior solution are:

$$\frac{u'(x_t)}{(1+\rho)^t} - \frac{\lambda}{(1+r)^t} = 0.$$

This implies that for two periods t' and t'', the tangency condition is:

$$\frac{u'(x_{t'})}{u'(x_{t''})} = \left(\frac{1+r}{1+\rho}\right)^{t''-t'}.$$

And, for two consecutive periods, $t'' = t' + 1$, this condition becomes:

$$\frac{u'(x_{t'})}{u'(x_{t'+1})} = \frac{1+r}{1+\rho}. \tag{4.3}$$

Armed with this tangency condition, we are prepared to ask the question, "Under what circumstances will consumption be increasing over time?"

Intuitively, what do you think the answer is? Hint: Consumption will be increasing over time if the consumer is (more or less) impatient than the market? What does it have to do with how r and ρ compare?

To make things simple, let's consider periods 1 and 2. The same analysis holds for any other two adjacent periods. By quasiconcavity of $U()$, we know that the consumer's indifference curves will be convex in the (x_1, x_2) space, as in Figure 4.6. When $x_1 = x_2$, the slope of the utility isoquant is given by $-\frac{u'(x_1)}{u'(x_2)}(1+\rho) = -(1+\rho)$. When $x_1 > x_2$, this slope is less than $(1+\rho)$ in absolute value. When $x_2 > x_1$, this slope is greater than $(1+\rho)$ in absolute value. The tangency condition (4.3) says that the absolute value of the slope of the isoquant must be the same as $(1+\rho)$. Thus if $1+r > 1+\rho$ (which is equivalent to $r > \rho$), the optimal consumption point must have $x_2 > x_1$: Consumption rises over time. If $1+r < 1+\rho$ (which is equivalent to $r < \rho$), $x_1 > x_2$, and consumption falls over time. If $1+r = 1+\rho$, consumption is constant over time.

What is the significance of the comparison between ρ and r? Starting from the situation where consumption is equal in both periods, the consumer is willing to give up 1 unit of future consumption for an additional $\frac{1}{1+\rho}$ units of consumption today. By giving up one unit of future consumption, the consumer can buy an additional $\frac{1}{1+r}$ units of consumption today.

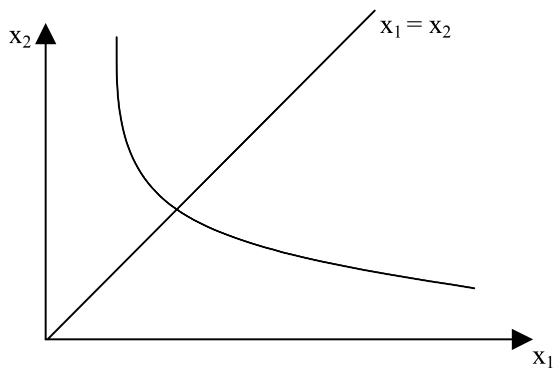

Figure 4.6: Two-Period Consumption

Thus if $\frac{1}{1+r} > \frac{1}{1+\rho}$, the consumer is willing to give up this unit of future consumption: Optimal consumption decreases over time. This condition will hold whenever $\rho > r$. On the other hand, if

$$\begin{aligned} \frac{1}{1+r} &< \frac{1}{1+\rho} \\ \rho &< r, \end{aligned}$$

the consumer would rather shift consumption into the future: Optimal consumption rises over time. In words, if you are more patient than the market, consumption tends to grow over time; but if you are less patient than the market, consumption tends to shrink over time.

4.6.4 The Fisher Separation Theorem

Suppose that the consumer must choose between two careers. Career A yields endowment vector $a = (a_0, ..., a_T)$. Career B yields endowment vector $b = (b_0, ..., b_T)$. Which should the consumer choose? One is tempted to think that in order to decide you have to solve the consumer's UMP for the two endowment vectors and compare the consumer's utility in the two cases. A remarkable result demonstrated by Irving Fisher, known as the Fisher Separation Theorem, shows that if the consumer has free access to credit markets, then the consumer should choose the endowment vector that has the largest present value. Put another way, the consumer's production decision (which endowment vector to choose) and her consumption decision (which consumption vector to choose) are separate. The consumer maximizes utility by first choosing the endowment vector with the largest present value and then choosing the consumption vector that maximizes utility, subject to the budget constraint implied by that endowment vector.

First, we need to explain what is meant by free access to credit markets. Basically, this means that the consumer can borrow or lend as much wealth as she wants at interest rate r, as long as her budget balances over the entire time horizon of the model. That is, all consumption vectors such that

$$\sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \leq \sum_{t=0}^{T} \frac{a_t}{(1+r)^t}$$

are available to the consumer.

The Fisher Separation theorem follows as a direct consequence of this. Let $PV_a = \sum_{t=0}^{T} \frac{a_t}{(1+r)^t}$ and $PV_b = \sum_{t=0}^{T} \frac{b_t}{(1+r)^t}$. The consumer's UMP for endowments a and b are given by:

$$\begin{aligned} &\max_x U(x_0, ..., x_T) \\ s.t &: \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \leq PV_a \end{aligned}$$

and

$$\begin{aligned} &\max_x U(x_0, ..., x_T) \\ s.t &: \sum_{t=0}^{T} \frac{x_t}{(1+r)^t} \leq PV_b. \end{aligned}$$

These problems are identical except for the right-hand side of the budget constraint. And, since we know that when utility is locally non-satiated, utility increases when the budget constraint is relaxed, so the consumer will achieve higher utility by choosing the endowment vector with the higher present value. It's that simple.

When the credit markets are not complete, the separation result will not hold. In particular, if the interest rate for saving is less than the interest rate on borrowing (as is usually the case in the real world), then the opportunities available to the consumer will depend not only on the present value of her endowment but also on when the endowment wealth is received. For example, consider Figure 4.7. Here, the interest rate on borrowing, r, is greater than the interest rate on saving, R. Because of this, beginning from initial endowment a, the budget line is flatter when the consumer saves (moves toward higher future consumption) than when she borrows (moves toward higher present consumption). Figure 4.7 depicts the budget sets for two initial endowments, a and b. Since neither budget set is included in the other, we cannot say whether the consumer prefers endowment a or endowment b without solving the UMP.

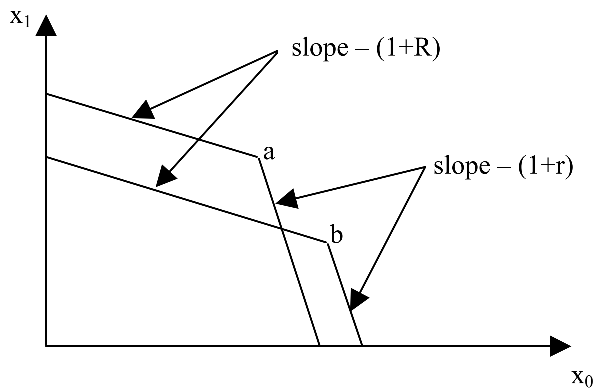

Figure 4.7: Imperfect Credit