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Loss Sequencing in Banking Networks: Threatened Banks as Strategic Dominoes *

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Abstract

We demonstrate in a stylized banking network that a single large loss has the potential to leave markedly different impacts on the financial system than does a sequence of moderate losses of the same cumulative magnitude. Loss sequencing matters because banks make strategic bailout decisions based on their myopic assessment of losses, yet these decisions are highly consequential to subsequent decisions and eventual losses at other banks in the network. In particular, the network mechanism enables banks to choose to bail out their creditors after every moderate loss incurred in a sequence, while walking away from the creditors should they experience a single large loss. Government policy can force threatened banks to liquidate or sell themselves or, at the opposite pole, can bail out some such banks or overlook their threatened status. The former policy would concentrate a string of losses into a single large event; the latter could prevent a massive single loss at the expense of multiple subsequent smaller losses. As this analysis shows, either policy could prove optimal depending on identifiable circumstances. These findings have important implications for on-going policy debates that emanated from the 2008 meltdown.

JEL-Classification: G21, G01, G28, G33.

Keywords: Cascades, Bailouts, Liquidations, Loss Sequence.

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1 Introduction

This paper raises and investigates the following question: Are financial networks more resilient to (i) a single large loss or (ii) a sequence of moderate losses of equivalent cumulative magnitude? A detailed analysis into this question, which includes identifying circumstances where alternative answers apply, has highly relevant implications for the design, management and regulation of a robust banking sector. It will inform the industry and regulators to concentrate resources and policies in strengthening the banking sector against the loss configuration under which the sector is more vulnerable – i.e., rationalizing the "too big to fail" or otherwise arguments. To invoke a metaphor, our comparison is between an earthquake (simultaneous losses) and a cascade of tumbling dominoes (sequential losses) where the cumulative losses are the same. The ultimate consequences differ because some dominoes can make strategic choices contingent on having observed how spectacularly dominoes ahead of them have tumbled, with the prime objective of preventing their demise and, given that, limiting their own losses. Such market-based coordination is simply not feasible for an earthquake event, absent government intervention.

We address this question in a stylized model of the banking network, which generalizes the Eisenberg and Noe (2001) (E-N) financial network – an emerging framework to investigate loss contagions via interconnectedness. Our main findings are; A single large loss leaves markedly different impacts on the financial system than does a sequence of moderate losses of the same cumulative magnitude. In particular, several moderate shocks hitting a single bank sequentially may entail a smaller total loss to the network than does a single large shock hitting the same bank. In addition, several moderate shocks hitting several banks simultaneously may entail a smaller total loss to the network than when they hit these banks sequentially. In sum, our study identifies stylized economic channels in which two polar opposite government policies – collapsing a string of losses into a single large event, and transforming a massive loss into a sequence of lesser losses – are desirable. The analysis has significant implications for government bailout strategy and regulatory policy, and therefore, contributes to important on-going debates on reforming and regulating financial industries.

Drawing on hindsight of the Great Recession of 2008-2010, our study is motivated by two remarkable events leading to the turmoil in financial markets of that period. (Detailed accounts and references are relegated to Appendix A) The first event recounts the active roles taken by the Fed in synchronizing (i) Lehman Brothers' bankruptcy filing and (ii) Bank of America acquiring Merrill

Lynch announcement into a single date. The second event recounts Bear Stearns' extraordinary bailout pledge of several billions of USD for its hedge fund – an unprecedented decision attributed to an attempt to avert an even larger Bear Stearns' loss of reputation (at other banks) amid huge mortgage business devaluation in 2007. These monumental events present us with important insights that (i) regulators and government agencies do perceive and proactively act on the relevance of loss sequencing[1] to the welfare of the financial system, and (ii) financial institutions strategically respond to their losses using strategies that vary with loss sizes and perceived market conditions.

Our modeling of the loss sequencing and its perception by market players in the financial system are informed by these realistic features. Our investigation starts with the original E-N financial network.[2] We prove a surprising analytical irrelevance result that the loss sequencing (i.e., how losses are partitioned intertemporally), has absolutely no effects on any constituent banks in the original E-N financial network. Once observed, this key result is intuitive. Whereas the market-clearing mechanism underlying the E-N setting incorporates banks' fundamental financing characteristics,[3] it features no active and contingent engagements of either banks or regulators towards losses of various sizes. Moreover, all debt holders have equal seniority in the E-N setting. As a result, even when a bank defaults, its creditors still recoup a proportional amount of their loans and the relative liability structure of the default bank stays intact. Therefore, while the impacts to the financial network and individual banks vary passively with the loss size, these impacts simply add up proportionally as losses add up. As a result, the final impact on the network depends solely on the cumulative size of all losses. The way that these losses are sequenced does not matter.

Our analysis centers on the recognition that market participants react strategically, albeit myopically, to losses. The relevance of the loss sequencing on the financial system then arises precisely when these players respond in this fashion to losses. We therefore enrich the original E-N setting with these realistic features, which also have their roots in the complexity of the financial network. Stylized by the accounts of Lehman Brothers' bankruptcy filings and the Bear Stearns' bailout mentioned earlier, we model banks' assets as their (equity) holdings in hedge funds, broadly construed, under their management. The hedge funds make bets on risky financial derivatives, e.g., mortgage-back securities. Funds are highly leveraged because they are financed largely by outside

[1]Here and throughout this paper, "loss sequencing" refers to a process in which losses are partitioned/distributed intertemporally, possibly through regulatory measures/actions.

[2]The Eisenberg-Noe (E-N) network setting offers an analytical framework to accommodate an arbitrary number of banks endowed with arbitrary interconnectedness, yielding unique market-clearing solution ex-post (after losses), and hence is our starting point.

[3]These characteristics are debt holders' seniority and equity holders' limited liability in case of defaults.

lenders, with collateral being the assets of the funds themselves.[4]

When bets turn sour at a hedge fund, its collateral goes under, and the sponsoring bank in our stylized model may either (i) post additional collateral to bail out the fund's lenders (thus "stay" with its non-binding sponsored debt obligations, i.e., implicit guarantees) or (ii) "walk away" by capitalizing on its limited liability protection, implying that the lenders bear losses from the devalued collateral. In the "stay" option, the sponsoring bank internalizes the losses of its hedge funds and the liquidation cost resulting from raising cash to cover collateral shortfalls. The lenders are effectively bailed out and thus suffer no losses. In the "walk away" option, the sponsoring bank internalizes a reputation loss.[5] This action however deals a negative externality first to the fund's lenders, who have no option but to seize and sell off their now underwater collateral. These are fire sales imposing a negative externality on to all other banks holding similar assets in the network.

A key stylized feature, which makes loss sequencing relevant in our model, is that banks make their choices based on their self-centered strategic perspective.[6] Strategically, the "stay" choice is preferred to "walk-away" choice for a bank if the reputation loss exceeds the combination of the direct loss to the bank's hedge fund and the liquidation cost, and vice versa.[7] Myopically, these reputation and liquidation losses, the basis on which banks make their choices, are the bank's ex-ante estimates (omitting network feedback effects). Because (i) banks make different self-centered choices when faced with different (sequential versus simultaneous) loss configurations, and (ii) eventual (ex-post) losses are endogenous to the choices banks made earlier,[8] the loss sequencing is relevant to the eventual equilibrium (clearing) state of the financial system.

Loss sequencing effects differ dramatically depending whether losses hit a single bank or several banks in the network. When losses hit only a single bank in the network, our finding that sequential losses entail a smaller total loss to the system than does a single large loss of the same cumulative magnitude is intuitive. Pursuing its self interest, a bank may choose to stay with every moderate loss in a sequence, yet walk away immediately from a single large (cumulative) loss. As a result, fire sales are avoided in the sequential loss configuration, sparing other banks from a devaluation

[4]This highly-leveraged business model was ubiquitous in the years leading to the Great Financial Crisis. Such large borrowing was possible because banks had "reputations", which "assured" lenders with ill-defined "implicit guarantees".

[5]As the bank backs away from its non-binding sponsored debt obligation.

[6]These choices are myopic because banks do not have crystal balls, and virtually cannot calculate network feedback effects due to network complexity and information asymmetry.

[7]In our setting, the liquidation cost is increasing with the original loss because the latter is proportional to the amount of (other) assets the bank need to liquidate to make up for the collateral shortfalls.

[8]Eventual (ex-post) losses are determined after banks have made their (ex-ante) choices (stay with or walk away from their sponsored debt obligation) towards original (ex-ante) losses and then markets have cleared in E-N style.

of their similar collateral assets, and therefore, improving welfare.

When losses hit several banks in the network, our finding that simultaneous losses entail a smaller total loss to the system than do sequential losses is also intuitive. When losses hit multiple banks sequentially, a second bank (i.e., the one that is hit with a later loss) makes a choice based on the observed choice made by the first bank (i.e., the one that was hit with an earlier loss). This hindsight advantage was not available when losses hit these banks simultaneously. In case the first bank has already walked away from the earlier loss, the second bank has less "skin in the game" (due to the fire sale associated with the first bank's default), and thus may also choose to walk away from the later loss. By contrast, the second bank otherwise would have chosen to stay with its loss if this loss had occurred simultaneously with the first bank's loss. Fire sales are more extensive in the sequential-loss configuration due to defaults at both banks. Therefore, welfare is lower than in the simultaneous-loss configuration. In our stylized model, this differentiated eventual loss to the total system in the two loss configurations gives a rationale for the Fed's active timing of the Lehman Brothers' bankruptcy filing and the announcement of Bank of America's acquisition of Merrill Lynch.

Related Literature

Our paper is related to several vibrant strands of literature studying the effects of losses, contagions and stability in the financial system, starting with the seminal work of Diamond and Dybvig (1983). In the equilibrium setting of the financial system, Allen and Gale (2000) show that the incompleteness of interbank claims is a key conduit of systemic loss contagions. Allen et al. (2012) have banks swapping opaque investment projects to diversify idiosyncratic risks, and they show that systemic contagions depend crucially on the maturity structure of debts. In Giglio (2014), spreads of the credit default swap written by a bank as an insurance on another bank's default are informative about, and hence are employed to estimate, the systemic (joint) default risk of these banks and the financial network. In Di Maggio and Tahbaz-Salehi (2014), collateralized borrowing among banks generates a financial network, which in turn is highly fragile and sensitive to the quality of collateral. In Babus (2016), banks face potential contagious losses as they commit ex-ante to mutually insure each other against aggregate liquidity shocks. Nevertheless, there exist equilibria in which contagions do not occur. In Gofman (2016), a non-monotonic tradeoff is shown between the interconnectedness, efficiency and stability of a network of financial institutions in a quantitative analysis. Our paper focuses on how the financial system's ex-post losses vary with the temporal distribution of loss arrivals. Our study highlights the possible role of loss sequencing in limiting systemic contagions.

We rely on the market-clearing mechanism in the financial network to characterize losses. With regard to the market-clearing mechanism, Eisenberg and Noe (2001) lay out a basic network structure involving an arbitrary number of financial entities and interbank linkages. They relate the market-clearing state of the network to the fixed-point of a contracting map, and hence derive its uniqueness. Financial contagions via E-N network interconnectedness are studied in Elliott et al. (2014), Acemoglu et al. (2015), and Cabrales et al. (2016) among others. It is found that higher (resp., lower) interconnectedness makes an E-N network more robust against small (resp., large) losses, which agrees with Gai and Kapadia (2010)'s simulation results showing that the financial system exhibits a robust-yet-fragile tendency. Glasserman and Young (2015) obtain upper bounds on the potential losses due to network contagion, and find these bounds to be quite limited.[9] The original E-N network has been generalized in several aspects. These aspects include equity cross-holdings by Elsinger (2011), default costs by Rogers and Veraart (2012), inefficient collateral liquidation upon defaults by Oehmke (2014), fire sale and capital requirement by Feinstein and El-Masri (2015), endogenous formation of linkages by Farboodi (2015) and Wang (2015). Our paper complements this growing literature by modeling banks' self-centered responses to their direct losses, a realistic feature that is key to understanding our quest: the importance of the loss sequencing in financial networks.

There is also an emerging literature on aspects of information contagion in the financial network. In Acharya and Thakor (2014), when a bank is liquidated, creditors at other banks update their beliefs about the systemic loss and inefficiently liquidate their assets. Our setting features information asymmetry (between banks suffering direct losses and the rest of the network), but does not model Bayesian updating of beliefs. For an overview of this and other recent developments, we refer to excellent review papers by Cabrales et al. (2015) and Glasserman and Young (2016)

Our paper is structured as follows. Section 2 reviews the original Eisenberg-Noe setting and derives a key analytical result that loss sequencing is irrelevant in that setting. Section 3 motivates and presents stylized features to enrich the E-N banking network. Section 4 demonstrates the relevance of loss sequencing when losses hit a single bank. Section 5 demonstrates the relevance of

[9] In this regard, modeling banks' strategic and myopic responses to losses as in our setting offers a realistic venue to revise these upper bounds significantly upwards.

loss sequencing when losses hit multiple banks. Section 6 concludes. Appendices A and B present supporting materials and derivations that are omitted in the main text.

2 Irrelevance of Loss Sequences in Financial Systems

We study the Eisenberg and Noe (2001) (E-N) network setting of financial systems to address the differential ability of financial systems to absorb a sequence of moderate losses as opposed to a single substantial loss of the same cumulative amount. Although the E-N network is a stylized setting, it offers an analytical and highly flexible framework to investigate the loss impact on an arbitrary set of interconnected financial entities (banks, firms, institutions). For self-sufficiency purposes, we first present the working setup of this type of financial network. We subsequently derive properties concerning the dynamics and clearing state when a sequences of losses hits this financial network. We establish a surprising irrelevance result (Theorem 2.2) that the loss sequencing in no ways affects the final (clearing-state) outcomes to financial entities in E-N-type networks. We also illustrate this result with a numerical example.

2.1 Eisenberg-Noe Network of Financial Systems

Eisenberg-Noe Setup

Our notation largely follows that of Eisenberg and Noe (2001) and Glasserman and Young (2015). In particular, in what follows, the bar denotes ex-ante values (i.e., before losses hit systems), and boldface denotes quantities in vector or matrix forms. Consider an economy with n financial firms (henceforth, banks) indexed by $i \in \mathcal{N} = \{1, \ldots, n\}$. Each bank i has payments $\overline{p}_{ij} \geq 0$ (henceforth, inside debts) due to respective banks $j \in \mathcal{N}$ in the system. Together, these debts form an $n \times n$ liabilities matrix $\overline{\boldsymbol{P}} = \{\overline{p}_{ij}\}$. We denote by an $n \times 1$ vector $\boldsymbol{c} = (c_1, \ldots, c_n)^T \geq \mathbf{0}$ the aggregate of all assets held by the banks. This section treats c_i as the collective value of bank i's outside assets and ignores their structure.[10] Furthermore, each bank i may have liability $b_i \geq 0$ (henceforth, outside debts) to entities outside the financial network. This E-N financial network can have arbitrary connectedness, in which $\overline{p}_{ij} = 0$ if i has no financial obligation to j, and $b_i = 0$ if i has no creditors outside of the network.[11] As in the original E-N setting, we assume that all liabilities have equal

[10] Banks' inside assets are accounted by elements of matrix $\overline{\boldsymbol{P}}$.

[11] Otherwise, when $\overline{p}_{ij} > 0$, when there is a financial linkage directed from i to j before losses hit the financial system: i is ex-ante debtor of j, or equivalently j is ex-ante creditor of i.

priority, an assumption that will be relaxed in later sections. Let $\overline{p}_i$ denote the aggregate obligations of bank i, and $\overline{\boldsymbol{p}} \in (\boldsymbol{R}^{+})^n$ denote the $n \times 1$ obligation vector in the network,

$$\overline{p}_i = \sum_{j=1}^{n} \overline{p}_{ij} + b_i, \qquad \overline{\boldsymbol{p}} = (\overline{p}_1, \ldots, \overline{p}_n)^T. \tag{1}$$

Let $\boldsymbol{A} = \{a_{ij}\}$, $i, j \in \mathcal{N}$ denote the $n \times n$ relative liabilities matrix, which is defined as follows,

$$a_{ij} = \begin{cases} \frac{\overline{p}_{ij}}{\overline{p}_i} & \text{if } \overline{p}_i > 0, \\ \\ 0 & \text{otherwise,} \end{cases} \qquad i, j \in \mathcal{N}. \tag{2}$$

Therefore, a_{ij} is the bank i's obligation to bank j, expressed as a fraction of bank i's total obligation $\overline{p}_i$ (1) to both entities inside and outside the network. The fraction of i's outside debts then is,

$$\frac{b_i}{\overline{p}_i} = 1 - \sum_{j}^{n} a_{ij}, \quad i, j \in \mathcal{N}, \quad \text{or} \quad \boldsymbol{A}^T \overline{p} + b = \overline{p}. \tag{3}$$

When i and j have no financial linkages, $a_{ij} = 0$ identically. The financial system can thus be unambiguously described by the set $\{\boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}\}$.

Let $n \times 1$ vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T \in (\boldsymbol{R}^{+})^n$ denote the losses to the assets of the banks which reduce their values to $\boldsymbol{c} - \boldsymbol{x} = (c_1 - x_1, \ldots, c_n - x_n)^T$. Following Glasserman and Young (2015), we assume that banks have limited liabilities with respect to their outside asset holdings, i.e., the size of losses never exceeds those assets' values,

$$0 \leq x_i \leq c_i, \quad \forall i \in \mathcal{N}, \qquad \text{or} \qquad \boldsymbol{c} - \boldsymbol{x} \geq \boldsymbol{0}. \tag{4}$$

Losses to the network otherwise have an arbitrary structure, with $x_i = 0$ conventionally signifying that bank i is not hit by any external losses.

Clearing Payment Vector

A key quantity in the E-N financial network is the $n \times 1$ clearing payment vector $\boldsymbol{p}(\boldsymbol{x})$, whose elements are aggregate obligations of banks in the network, resulting from the loss vector $\boldsymbol{x}$ hitting the system. Thus, the clearing payment vector is the ex-post counterpart of the ex-ante obligation vector $\overline{\boldsymbol{p}}$. The solution of the clearing payment vector follows from its defining properties.

Definition 1 *A clearing payment vector* $\boldsymbol{p}(\boldsymbol{x}) \in [\boldsymbol{0}, \overline{\boldsymbol{p}}]$ *for the financial system* $\{\boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}\}$ *resulting from losses* $\boldsymbol{x}$ *satisfies the following ex-post conditions:*

1. *Limited liability: Each bank i's obligations do not exceed its receivable cash flows,*

$$p_i(\boldsymbol{x}) \leq \sum_{j=1}^{n} a_{ji} p_j(\boldsymbol{x}) + c_i - x_i, \quad \forall i \in \mathcal{N}.$$

2. *Absolute debt priority: Each bank i's obligations are either paid fully or partially met by depleting i's receivable cash flows,*

$$p_i(\boldsymbol{x}) = \min\left\{\sum_{j=1}^{n} a_{ji} p_j(\boldsymbol{x}) + c_i - x_i \ , \ \overline{p}_i\right\}, \quad \forall i \in \mathcal{N}.$$

These conditions reflect the standard seniority hierarchy of claims between debts and equities. Bank i's creditors recover the full face value of debts when losses are moderate (i is solvent), and the full assets of i when losses are large (i defaults). In the latter case, an important feature of the E-N setting is that i's assets are divided among all (inside and outside) creditors of i on a pro-rata basis, since the relative liabilities matrix remains constant throughout. This implies an equal seniority of all debt claims on i from creditors inside and outside the network.[12]

Formally, the clearing payment vector $\boldsymbol{p}(\boldsymbol{x})$ is a fixed point of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) : [\boldsymbol{0}, \overline{\boldsymbol{p}}] \to [\boldsymbol{0}, \overline{\boldsymbol{p}}]$ defined by,

$$\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) = \min\{\boldsymbol{A}^T \boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x} \ , \ \overline{\boldsymbol{p}}\}. \tag{5}$$

This equation is the basis for the algorithm to solve and characterize the equilibrium outcomes of the ex-post liabilities and assets at every bank in an E-N network. By construction, $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ generates a non-increasing iterative liability sequence $\boldsymbol{p}^{t+1}(\boldsymbol{x}) = \Phi(\boldsymbol{p}^t(\boldsymbol{x}); \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ (see Appendix B.1). The liability sequence is bounded from below by $\boldsymbol{0}$, which guarantees the existence of fixed-point solutions $\boldsymbol{p}(\boldsymbol{x})$. A sufficient condition for the fixed point to be unique, originally established by Glasserman and Young (2015), is reproduced below for completeness.

Proposition 1 *There exists a unique clearing payment vector* $\boldsymbol{p}(\boldsymbol{x})$ *as result of a loss vector* $\boldsymbol{x}$ *hitting the E-N financial system, if from every bank i in the network there is a chain of positive*

[12]This is because in the E-N construction, i's ex-post debt to outside creditors, $b_i(\boldsymbol{x}) = p_i(\boldsymbol{x}) - \sum_j a_{ij} p_i(\boldsymbol{x})$, does not alter the ex-ante fraction of outside debt, $\frac{b_i(\boldsymbol{x})}{p_i(\boldsymbol{x})} = 1 - \sum_j^n a_{ij} = \frac{b_i}{\overline{p}_i}$.

(ex-ante) obligations to some bank j that has strictly positive obligations $b_j > 0$ to entities outside the financial network.[13]

A detailed proof of Proposition 1 is given in Appendix B.2. Technically, the uniqueness of the clearing payment vector is tantamount to the uniqueness of the fixed point of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ (5). Intuitively, when $b_j > 0$, bank j's inside debt fraction is strictly less than unity, or $\sum_k^n a_{jk} = (\boldsymbol{A}\,\mathbf{1})_j < 1$ by virtue of (2). Similarly, if bank i has a positive liability chain of length k to such bank j, then $\left(\boldsymbol{A}^k\,\mathbf{1}\right)_j < 1$. Therefore, if every bank i in the network has a positive liability chain to a bank j that has positive outside liability, then the relative liability matrix $\boldsymbol{A}$ must have spectral radius strictly less than unity.[14] Such spectral radius implies that $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ (5) is a contracting map, which then guarantees that the fixed point $\boldsymbol{p}(\boldsymbol{x})$ is unique. Note that whereas the positive liability chains referred to in Proposition 1 comprise only a sufficient condition for the uniqueness of the fixed point of the map (5), $\boldsymbol{A}$'s spectral radius being strictly less than unity is both a necessary and sufficient condition (Appendix B.2).

We observe, by construction of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$, that losses in vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ hit banks simultaneously. We next derive several properties of the E-N financial network to shed light on the important issue of whether and how loss sequences may lead to differential impacts on the financial system.

2.2 Eisenberg-Noe Setting: Irrelevance of the Loss Sequences

Within the original E-N network setting, the default on liability claims is the key factor that determines the relevance of the loss sequence on the financial system. On one end of the loss spectrum, when all losses $\{x_i\}$ in the loss vector $\boldsymbol{x}$ are sufficiently small, no defaults arise. Consequently, in this case, the map (5) effectively amounts to a linear operator, which generates no relevance of the loss sequence on the financial system. On the other end, a sequence of larger losses have more subtle consequences because it imposes defaults within the network. Our analysis starts with the following classification concerning defaults in the E-N framework.

Definition 2 *Ex-post, as result of (simultaneous) losses $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ to the financial system,*

[13] A chain of positive (ex-ante) obligations from bank i to bank j of length k is a sequence of k banks $\{i, i_1, \ldots, i_{k-1}, j\}$ such that all ex-ante liabilities $\overline{p}_{ii_1}$, $\overline{p}_{i_1 i_2}$, ..., $\overline{p}_{i_{k-1}k}$ are strictly positive. Equivalently, by virtue of (3), this means (i) all elements $a_{i_0 i_1}$, $a_{i_1 i_2}$, ..., $a_{i_{k-1}k}$ are strictly positive, and (ii) ex-ante aggregate debts of banks in the chain $\overline{p}_i$, $\overline{p}_{i_1}$, ..., $\overline{p}_{i_{k-1}}$ are strictly positive.

[14] Otherwise, $\boldsymbol{A}^k$ diverges as the exponent k increases, which contradicts $\left(\boldsymbol{A}^k\mathbf{1}\right)_j < 1$ for finite exponent k.

1. *Outright default: A bank i is in outright default, $i \in \mathcal{OD}(\boldsymbol{x})$, if,*[15] $\sum_j^n a_{ji} p_j(\boldsymbol{x}) + c_i - x_i = 0$.

2. *Default: A bank i is in default, $i \in \mathcal{D}(\boldsymbol{x})$, if,* $0 < \sum_j^n a_{ji} p_j(\boldsymbol{x}) + c_i - x_i < \overline{p}_i$.

3. *Solvent: A bank i is solvent, $i \in \mathcal{S}(\boldsymbol{x})$, if,* $0 < \overline{p}_i < \sum_j^n a_{ji} p_j(\boldsymbol{x}) + c_i - x_i$.

Note that in the cases i is in outright default or default, the cash flows in and the cash flows out at bank i exactly offset each other, reducing its equity value to zero. Resulting from an arbitrary (and simultaneous) loss vector $\boldsymbol{x}$, banks in the financial system are classified into three mutually exclusive sets,

$$\mathcal{N} \;=\; \mathcal{OD}(\boldsymbol{x}) \;\cup\; \mathcal{D}(\boldsymbol{x}) \;\cup\; \mathcal{S}(\boldsymbol{x}). \tag{6}$$

We notice that in the case of defaults, the redistribution of debt claims on a pro-rata basis featured in the E-N framework leads to a clear-cut separation between banks in outright default and the rest. Those in outright default must have all their assets and receivable accounts wiped off; those not in outright default must have all their payable obligations remain strictly positive (albeit, possibly written down) by losses. These simple "make it or break it" necessary and sufficient conditions for outright defaults are as follows.

Proposition 2 *In the E-N setting, following losses in $\boldsymbol{x}$,*

1. *a bank $i \in \mathcal{N}$ is in outright default if and only if these losses wipe out (i) the entire external assets of i, and (ii) all internal debts receivable by i,*

$$i \in \mathcal{OD}(\boldsymbol{x}) \iff \begin{cases} c_i - x_i = 0, \\ p_{ji}(\boldsymbol{x}) = 0, \; \forall j \in \mathcal{N}. \end{cases} \tag{7}$$

2. *a bank $i \in \mathcal{N}$ is not in outright default if either (i) the external assets of i remain strictly positive (if i has external assets initially), or (ii) some internal obligations payable by i remain strictly positive,*

$$i \notin \mathcal{OD}(\boldsymbol{x}) \iff \left[\begin{array}{l} c_i - x_i > 0, \quad \text{if } c_i > 0, \\ \exists k \text{ s.t. } p_{ik}(\boldsymbol{x}) > 0, \quad \text{and } a_{ik} > 0. \end{array} \right. \tag{8}$$

[15] By virtue of the limited liability assumption (4), $\sum_j^n a_{ji} p_j(\boldsymbol{x}) + c_i - x_i \geq 0$, which facilitates the classification of outright default.

These conditions for outright defaults are intuitive. Because debt claims are written down on a pro-rata basis, a bank i is in outright default if and only if all of its payable inside debts $\{p_{ik}(\boldsymbol{x})\}_k$, and all of its outside debt $b_i(\boldsymbol{x})$ are written down to zero altogether. For this to happen, i must have no receivable in cash flows, either from inside or outside the network,[16] after losses $\boldsymbol{x}$ hitting the system, i.e., conditions (7). Therefore, a bank in outright default has neither ex-post in nor out cash flows in the E-N framework. From this intuition, a further necessary condition for an outright default can be obtained from Proposition 2.

Proposition 3 *A bank $i \in \mathcal{N}$ being in outright default necessarily implies that all debtors k of i must also be in outright default,*

$$i \in \mathcal{OD}(\boldsymbol{x}) \implies k \in \mathcal{OD}(\boldsymbol{x}), \quad \forall k \quad \textit{such that} \quad a_{ki} \neq 0.$$

Sequential versus Simultaneous Losses

To investigate the impact of the loss sequence on the financial network, consider two loss vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. We denote by $\boldsymbol{x}{\to}\boldsymbol{y}$ the sequence of losses in which losses in $\boldsymbol{x} = (x_1, \dots, x_n)^T$ are realized first, and then those in $\boldsymbol{y} = (y_1, \dots, y_n)^T$ are realized second. When losses in $\boldsymbol{x}$ and $\boldsymbol{y}$ happen simultaneously, we write $\boldsymbol{x}{+}\boldsymbol{y}$. Accordingly, let $\boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y}) \in (\mathbf{R}^+)^n$ denote the clearing payment vector after sequential losses $\boldsymbol{x}{\to}\boldsymbol{y}$. Therefore, $\boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y}) \in (\mathbf{R}^+)^n$ is also the fixed point of a sequence of two maps of type (5),

$$\begin{aligned} \boldsymbol{p}(\boldsymbol{x}) &= \Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) &&= \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x}\ ,\ \overline{\boldsymbol{p}}\}, \\ \boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y}) &= \Phi(\boldsymbol{p}; \boldsymbol{A}, \boldsymbol{p}(\boldsymbol{x}), \boldsymbol{c} - \boldsymbol{x}, \boldsymbol{b}, \boldsymbol{y}) &&= \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x} - \boldsymbol{y}\ ,\ \boldsymbol{p}(\boldsymbol{x})\}. \end{aligned} \tag{9}$$

We first observe that the same sufficient condition underlying Proposition 1 yields a unique fixed point for the above sequence of maps. Indeed, assume the Glasserman and Young (2015) condition that before any losses occur, from every bank i in the network, there is a chain of positive obligations to some bank j that has strictly positive obligations $b_j > 0$ to entities outside the financial network. Then let the loss vector $\boldsymbol{x}$ hit the system, and $\mathcal{OD}(\boldsymbol{x})$ be the unique set of banks in outright default as a results of these losses. The uniqueness of $\mathcal{OD}(\boldsymbol{x})$ and the clearing vector $\boldsymbol{p}(\boldsymbol{x})$ flows from Proposition 1. Moreover, by virtue of Proposition 3, a bank k that is not in outright default (i.e.,

[16]By the assumption of limited liability (4), the largest possible external loss to bank i is its outside assets, $x_i = c_i$. Effectively, an external loss exceeding c_i is written down to c_i. This maximum external loss is a necessary condition for i to be in outright default.

$k \in \mathcal{N} \setminus \mathcal{OD}(\boldsymbol{x})$) cannot have a positive liability chain to any outright default bank $i \in \mathcal{OD}(\boldsymbol{x})$.[17] Therefore, for the above Glasserman and Young (2015) condition to hold initially, every bank $k \in \mathcal{N} \setminus \mathcal{OD}(\boldsymbol{x})$ must have some (ex-ante) chain of positive obligations to some bank j with $b_j > 0$. As a result of Proposition 2, the loss vector $\boldsymbol{x}$ can neither alter nor break the positivity characteristic of any of these chains. Therefore we can apply Proposition 1 one more time on the second map in (9) to obtain the uniqueness for the clearing payment vector $\boldsymbol{p}(\boldsymbol{x}\rightarrow\boldsymbol{y})$. This result is recapitulated in the next proposition, which extends Proposition 1 to the setting of sequential losses.

Proposition 4 *There exists a unique clearing payment vector $\boldsymbol{p}(\boldsymbol{x}\rightarrow\boldsymbol{y})$ as result of a loss vector $\boldsymbol{x}$ hitting the E-N financial system, if from every bank i in the network there is a chain of positive (ex-ante) obligations to some bank j that has strictly positive obligations $b_j > 0$ to entities outside the financial network.*

The standard feature that debt claims can only total up to the face value of the debt immediately implies that as losses sequentially hit the financial system (9), the clearing payment vector is non-increasing,

$$\begin{array}{rcccl} \boldsymbol{p}(\boldsymbol{x}) & = & \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x}\ ,\ \overline{\boldsymbol{p}}\} & \leq & \overline{\boldsymbol{p}}, \\ \boldsymbol{p}(\boldsymbol{x}\rightarrow\boldsymbol{y}) & = & \min\{\boldsymbol{A}^T\boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{c} - \boldsymbol{x} - \boldsymbol{y}\ ,\ \boldsymbol{p}(\boldsymbol{x})\} & \leq & \boldsymbol{p}(\boldsymbol{x}). \end{array} \tag{10}$$

Because the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ is non-increasing in the initial-debt argument $\overline{\boldsymbol{p}}$ (Appendix B.1), the above relationships immediately imply that,

$$\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y}) = \Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}-\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{y}) \ \geq\ \Phi(\boldsymbol{p}; \boldsymbol{A}, \boldsymbol{p}(\boldsymbol{x}), \boldsymbol{c}-\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{y}) = \boldsymbol{p}(\boldsymbol{x}\rightarrow\boldsymbol{y}) \ \geq\ \boldsymbol{0},$$

or the clearing payment vector after sequential losses cannot exceed that vector after simultaneous losses. Consequently, if bank i is in outright default after simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$, or $p_i(\boldsymbol{x}+\boldsymbol{y}) = 0$, then it must be that $p_i(\boldsymbol{x}\rightarrow\boldsymbol{y}) = 0$, or that i is also in outright default after sequential losses $\boldsymbol{x}\rightarrow\boldsymbol{y}$. Similarly, if bank i is in default after simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$, or $p_i(\boldsymbol{x}+\boldsymbol{y}) < \overline{p}_i$, then it must be that $p_i(\boldsymbol{x}\rightarrow\boldsymbol{y}) < \overline{p}_i$, or i is also in default after sequential losses $\boldsymbol{x}\rightarrow\boldsymbol{y}$. Altogether, we have following result.

Proposition 5 *In the E-N setting, the adverse impact of sequential losses to the financial system*

[17] Otherwise, if k has a positive liability chain to a bank $i \in \mathcal{OD}(\boldsymbol{x})$, then k must also be in outright default $k \in \mathcal{OD}(\boldsymbol{x})$ as result of Proposition 3, which contradicts the assumption that k is not in outright default as a result of losses $\boldsymbol{x}$.

is at least as bad as that of the simultaneous losses of the same cumulative magnitude,

$$\mathcal{OD}(\boldsymbol{x}{+}\boldsymbol{y}) \subset \mathcal{OD}(\boldsymbol{x}{\to}\boldsymbol{y}), \qquad \mathcal{D}(\boldsymbol{x}{+}\boldsymbol{y}) \subset \mathcal{D}(\boldsymbol{x}{\to}\boldsymbol{y}), \qquad \mathcal{S}(\boldsymbol{x}{+}\boldsymbol{y}) \supset \mathcal{S}(\boldsymbol{x}{\to}\boldsymbol{y}).$$

In Proposition 5, the first two results concerning outright default and default banks immediately imply the last result concerning solvent banks. Intuitively, in the E-N framework, the adverse impact to a network ex-post (after losses) is commensurate with the amount of debt write-off. Therefore, a larger clearing payment vector signifies smaller default-related losses to the financial system, and thus fewer (more) banks in default (solvent) as evident in Proposition 5. In the case of simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$, the initial benchmark $\overline{\boldsymbol{p}}$ for ex-post debt claims is not lower than the corresponding benchmark $\boldsymbol{p}(\boldsymbol{x})$ in the case of sequential losses $\boldsymbol{x}{\to}\boldsymbol{y}$. Accordingly, the algorithm underlying the non-increasing map (5) mechanically generates a unique fixed point $\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y})$ that is not lower than the unique fixed point $\boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y})$ of the sequential maps (9).

Surprisingly, within the E-N setting of a financial network, the result also holds in the opposite direction, so that the fixed points of two simultaneous and sequential maps coincide. To frame this topological result in a metaphor, in the E-N framework it matters not whether a specified quantity of cumulative losses comes from an earthquake or a cascade of dominoes.

The next theorem summarizes this central result of the current section.

Theorem 1 (Irrelevance of loss sequencing in Eisenberg-Noe network) *In the E-N framework of a financial system, the sequence of losses is irrelevant to the resulting clearing payment vectors,* $\boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y}) = \boldsymbol{p}(\boldsymbol{x}{+}\boldsymbol{y})$*. Consequently, defaults and contagion structures in the E-N financial system do not depend on loss sequencing,*

$$\mathcal{OD}(\boldsymbol{x}{\to}\boldsymbol{y}) = \mathcal{OD}(\boldsymbol{x}{+}\boldsymbol{y}), \quad \mathcal{D}(\boldsymbol{x}{\to}\boldsymbol{y}) = \mathcal{D}(\boldsymbol{x}{+}\boldsymbol{y}), \quad \mathcal{S}(\boldsymbol{x}{\to}\boldsymbol{y}) = \mathcal{S}(\boldsymbol{x}{+}\boldsymbol{y}), \tag{11}$$

for generic loss vectors $\boldsymbol{x}$, $\boldsymbol{y}$.

The derivation of this irrelevance result is instructive, because it unveils further properties of the E-N setting. It also informs us of relevant features to model situations where loss sequencing is important.

Proof. The structure of the proof is intuitive. We first take clearing payments $p_i(\boldsymbol{x}\to\boldsymbol{y})$ (resulting from sequential losses $\boldsymbol{x}\to\boldsymbol{y}$) as given. They satisfy an explicit version of (9),

$$p_i(\boldsymbol{x}\to\boldsymbol{y}) = \min\left\{\sum_j^n a_{ji}p_j(\boldsymbol{x}\to\boldsymbol{y}) + (c_i - x_i) - y_i \ , \ p_i(\boldsymbol{x})\right\}, \qquad \forall i \in \mathcal{N}. \tag{12}$$

We then shown that, these payments $p_i(\boldsymbol{x}\to\boldsymbol{y})$ are a solution of the map associated with simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$, i.e., we will establish the following identity,

$$p_i(\boldsymbol{x}\to\boldsymbol{y}) = \min\left\{\sum_j^n a_{ji}p_j(\boldsymbol{x}\to\boldsymbol{y}) + c_i - (x_i + y_i) \ , \ \overline{p}_i\right\}, \qquad \forall i \in \mathcal{N}. \tag{13}$$

The uniqueness of the fixed point of two maps (one associated with sequential losses $\boldsymbol{x}\to\boldsymbol{y}$, and the other with simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$, Proposition 4) immediately renders the sought-after equality $\boldsymbol{p}(\boldsymbol{x}\to\boldsymbol{y}) \ = \ \boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y})$.

We first consider $i \in \mathcal{OD}(\boldsymbol{x}\to\boldsymbol{y})$. These banks i are in outright default after sequential losses hit the financial system. Therefore, by virtue of (12) and Proposition 2, we have,

$$p_j(\boldsymbol{x}\to\boldsymbol{y}) = 0, \qquad \sum_j^n a_{ji}p_j(\boldsymbol{x}\to\boldsymbol{y}) = 0, \qquad c_i = x_i + y_i, \qquad \forall i \in \mathcal{OD}(\boldsymbol{x}\to\boldsymbol{y}).$$

Given that $\overline{p} \geq \mathbf{0}$, the above equalities immediately imply (13) for all $i \in \mathcal{OD}(\boldsymbol{x}\to\boldsymbol{y})$.

Second, consider $i \in \mathcal{S}(\boldsymbol{x}\to\boldsymbol{y})$. These banks i are solvent after sequential losses hit the financial system, so they must have been solvent as well after first losses $\boldsymbol{x}$ (but before second losses $\boldsymbol{y}$) hit. Therefore, by virtue of Proposition 5, for all $i \in \mathcal{S}(\boldsymbol{x}\to\boldsymbol{y})$ we have $p_i(\boldsymbol{x}) = \overline{p}_i$. Consequently, (12) is identical to, and thus implies, identity (13) for all $i \in \mathcal{S}(\boldsymbol{x}\to\boldsymbol{y})$.

Third, consider $i \in \mathcal{D}(\boldsymbol{x}\to\boldsymbol{y})$. These banks i are in default after sequential losses hit the financial system. Therefore, by virtue of (12) we have,

$$\sum_j^n a_{ji}p_j(\boldsymbol{x}\to\boldsymbol{y}) + (c_i - x_i) - y_i < p_i(\boldsymbol{x}), \qquad \forall i \in \mathcal{D}(\boldsymbol{x}\to\boldsymbol{y}).$$

Because debt write-downs invariably progress with loss incidence, $p(\boldsymbol{x}) \leq \overline{p}$ (10), the above inequality implies,

$$\sum_j^n a_{ji}p_j(\boldsymbol{x}\to\boldsymbol{y}) + (c_i - x_i) - y_i < \overline{p}_i, \qquad \forall i \in \mathcal{D}(\boldsymbol{x}\to\boldsymbol{y}).$$

From this, identity (12) follows for all $i \in \mathcal{D}(\boldsymbol{x}\rightarrow\boldsymbol{y})$.

Since the three above sets exhaust the entire financial network (6), identity (13) indeed holds for all banks in the network. Equivalently, $p(\boldsymbol{x}\rightarrow\boldsymbol{y})$ – which by definition is the fixed point of the sequential maps (9) – is also the fixed point of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}+\boldsymbol{y})$ associated with simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$. Finally, because these maps have a unique fixed point, we have,

$$\boldsymbol{p}(\boldsymbol{x}\rightarrow\boldsymbol{y}) \;=\; \boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y}).$$

We also observe that the ex-post assets of every bank after sequential losses $(\boldsymbol{c}-\boldsymbol{x})-\boldsymbol{y}$ are identical to those after simultaneous losses $\boldsymbol{c}-(\boldsymbol{x}-\boldsymbol{y})$. Together with the above equalities concerning the clearing payment vectors, we conclude that the default and solvency structures of the financial system are also identical after sequential and simultaneous losses, which proves identities (11) ■

A Look Back at the Eisenberg-Noe Setting

The irrelevance of loss sequencing to financial systems, which is idealistically established for the E-N framework in Theorem 1 above, is surprising. It also appears at odds with real world observations. Central to the static E-N setting are two critical features, namely: (i) banks and the network passively absorb and redistribute losses and (ii) debt claims of all types enjoy equal seniority in the case of a default. The first feature is characterized by the mechanical market-clearing equation system (5), and the second by the constant nature of the relative liability matrix $\boldsymbol{A}$ (2). These simplifications deliver the analytical power of the original E-N setting. They also make loss sequencing irrelevant. Reality differs. Banks actively make strategic choices concerning asset reallocation following losses to their external businesses given asymmetric information and/or weak oversight.[18] Moreover, in practice, bank debt seniorities are far more complex than a flat structure. In the sections below, we enrich the original E-N setting to allow banks to make self-centered choices to minimize the adverse impacts of their own incoming losses. Such choices impose differentiated externalities on their creditors of different seniorities. Banks' decisions may also affect the entire network if they incur fire sales or inefficiently liquidate assets similar to those held by other banks. Positing such choices, we demonstrate next the relevance of loss sequencing for financial systems.

[18] Banks likely are aware of the under-performance or incoming losses of their loans to entities outside of the intermediaries network before general markets.

3 Financial Systems Enriched with Strategic Decisions

In this section, we extend the original E-N setting of financial networks. We present institutional details, make stylized assumptions, and discuss the practical relevance, of important new economic features that the extended setting includes.[19] We model the following features.

Implicit Guarantees and Bank Sponsored Debts

In our generalized setting, each bank i's asset holdings c_i can be considered i's sponsored business. Bank i funds these asset holdings c_i by (i) putting down some small initial capital amount f_i, and (ii) raising sizable debts of face value d_i from creditors (investors) $\mathcal{I}$ outside of the financial network,

$$\boldsymbol{c} = \boldsymbol{d} + \boldsymbol{f},$$

where $\boldsymbol{d}$, $\boldsymbol{f}$ are $n \times 1$ vectors of respective bank-level quantities d_i, f_i. In practice, banks' highly levered external businesses, $d_i \gg f_i$, are possible because banks use their reputations, as well as their substantial involvements in the financial network, to sponsor the debt raising d_i under some form of non-binding "implicit guarantees". Furthermore, assets $\boldsymbol{c}$ themselves are collateral in these sponsored businesses. With respect to the priority on this collateral, we assume that claims on sponsored debts $\boldsymbol{d}$ can be settled before other claims on both interbank liabilities $\boldsymbol{p}$ and other outside obligations $\boldsymbol{b}$. This feature can be motivated on the ground of information asymmetry, in which the directly-hit banks are first to know of losses in their sponsored businesses (as detailed below). This feature also aims to capture a practice that sponsored creditors $\mathcal{I}$ are exempted from automatic stays, and can seize the collateral assets $\boldsymbol{c}$ when banks default on their debts $\boldsymbol{d}$.

Bank also pursues investments in illiquid assets of initial book values $\boldsymbol{e} = (e_1, \dots, e_n)^T$. In case the sponsored businesses go under, banks may choose to liquidate these assets $\boldsymbol{e}$ to bail out their sponsored businesses. For simplicity, we assume that banks' illiquid asset holdings are commensurate with their highly-levered sponsored investments,

$$c_i = d_i = e_i, \qquad i \in \mathcal{N}. \tag{14}$$

This assumption purely aids tractability, and can be loosened to incorporate haircuts or partial

[19] The impacts of loss sequencing to the financial systems are demonstrated qualitatively and quantitatively in Sections 4 and 5.

collateral ($d_i < c_i$), as well as under-funded ($d_i > e_i$) or over-funded ($d_i < e_i$) implicit guarantees, without compromising the relevance of loss sequencing in the current generalized setting of financial systems. The illiquid nature of assets $\boldsymbol{e}$ aims to capture and quantify the reputations of banks.[20] Essentially, in our setting, banks' illiquid assets represent their "skin in the game", just as banks' capital serves that role in Holmstrom and Tirole (1998). They assure that banks stay with their obligations under limited liability, as we see next.

Losses and Margin Calls

Typical assets $\boldsymbol{c}$ in banks' sponsored businesses include mortgage-backed securities. In this case, the loss vector $\boldsymbol{x}$ records the decreases in the value of the underlying mortgages held by banks' sponsored businesses due to devaluation and defaults in that market. There is a clearing house, $\mathcal{C}$, which monitors the collateral values, $\boldsymbol{c}$ (with the debt values $\boldsymbol{d}$ being the margin-call benchmark). However, there is a hierarchy in the knowledge of the external losses (i.e., information asymmetry). If bank i's external assets suffer losses ($x_i > 0$), i is the first to become aware of its own losses.[21] Next, as soon as the clearing house learns of the drop in collateral values, it makes a margin call of the amount x_i (equal to the loss) to every bank i that suffers losses in assets. Having been aware of its own losses before anyone else in the system, bank i can choose to "stay" and meet the margin call (fulfilling its implicit guarantees) by posting additional collateral in the amount of x_i. Bank i can alternatively choose to "walk away" from its sponsored debt obligations, in which case the clearing house seizes the collateral on behalf of i's sponsored creditors $\mathcal{I}$. In our setting, walk-away amounts to a strategic (self-centered) default. Given the information-flow hierarchy mentioned above, this default represents an option to banks that are directly hit by external losses $\boldsymbol{x}$. Accordingly, the information asymmetry allows banks to reach settlements (either default or stay) with their sponsored creditors first, before interbank and other debt claims are addressed (the latter occur via market clearing). We refer to this feature as the "settlement priority" hereafter.

Self-centered Defaults

Though the collateral values $\boldsymbol{c}$ may drop below the face values $\boldsymbol{d}$ of the sponsored debts, banks do not necessarily walk away from their sponsored debt obligations because doing so damages their

[20] We observe that, in our setting, relaxing assumption (14) and exploring the effect of the ratio $\frac{d_i}{c_i} < 1$ on losses and defaults mirror the study of capital requirements on the welfare of the financial system.

[21] As we see next, this hierarchy creates rooms for bank i to make a strategic default decision in our setting.

reputations. Therefore, banks strategically consider whether to meet a margin call (henceforth, to "stay" or to "bail out" the sponsored business), or to break their implicit guarantees (i.e., to "walk away", meaning not to "bail out" the sponsored business).

On the one hand, staying requires bank i to sell a fraction η_i of its illiquid assets e_i to meet the margin call. Because these assets are illiquid, they can only be sold at a discount $\rho \in (0,1)$ (i.e., liquidation cost). Therefore, η_i solves margin call equation, $(1-\rho)e_i\eta_{xi} = x_i$. Bank i's illiquid asset accordingly drops as a result of this liquidation sale,

$$e_i \longrightarrow e_i - \eta_{xi} = \left(1 - \frac{x_i}{1-\rho}\right) e_i, \qquad \rho \in (0,1), \quad \forall i \in \mathcal{N}. \tag{15}$$

On the other hand, if it walks away, bank i incurs a reputation loss, which we model as a reduction in the marketable value e_i of i's illiquid assets as follows,

$$e_i \longrightarrow (1-\mu)e_i, \qquad \mu \in (0,1), \quad \forall i \in \mathcal{N}. \tag{16}$$

Here $\mu \in (0,1)$ is a measure of the reputation damage a bank suffers. For simplicity, we have assumed that μ is the same for all banks and is public information. This reputation loss played a key role to prompt Bear Stearns' (ultimately, insufficient and unsuccessful) decision to bail out its two subsidized funds in 2007, even when losses had started looming very large over the subprime mortgage business of these funds and over the industry more generally.

Because of the financial network's complexity, a bank makes decisions based on its self-centered (subjective yet myopic) valuation of the cost associated with the stay or walk-away options. If bank i stays after losses $\boldsymbol{x}$, its perceived cost includes the direct loss x_i and the liquidation cost $\rho e_i \eta_{xi}$ stemming from meeting margin calls, or $x_i + \rho e_i \eta_{xi} = \frac{x_i}{1-\rho}$ (15). If bank i walks away after losses, the cost is its reputation damage μe_i. After comparing these two costs, bank i will opt for a self-centered default (breaking its implicit guarantees) and walk away from its sponsored debt obligations if, $\frac{x_i}{1-\rho} \geq \mu e_i$. Otherwise, bank i will opt to stay (fulfilling its implicit guarantees) and meet the margin call by liquidating a part of its illiquid assets. Therefore, by virtue of (14), the self-centered decision-making protocol for every bank i is as follows,

$$\begin{array}{lll} \text{If} & x_i \geq \mu(1-\rho)e_i & \text{walk away from sponsored debt obligations.} \\ \text{Otherwise} & x_i < \mu(1-\rho)e_i & \text{stay and fulfill implicit guarantees.} \end{array} \tag{17}$$

Intuitively, bank i strategically chooses to walk away from its sponsored debt obligations d_i when either (i) the external direct loss outweighs its reputation damage (large x_i and/or small μ) or (ii) the liquidation is costly (large ρ). In the former case, a severe loss renders little residual value $c_i - x_i$ for i's sponsored assets, making it undesirable for i to stick with these assets by posting additional collateral of sizable value $\frac{x_i}{1-\rho}$.

Fire Sales

When bank i opts for a self-centered default and walks away from its sponsored debt obligations, it imposes a negative externality on the financial system because its collateral assets must be disposed through a fire sale (e.g., Kiyotaki and Moore (1997)). We assume that impatient sponsored creditors $\mathcal{I}$ cannot keep the seized collateral on their balance sheet after i's self-centered default. Rather, they unload the collateral on markets just when these assets have suffered losses $\boldsymbol{x}$. This sell-off causes a decline (i.e., fire sale) in asset value from c_k to $(1-\lambda)c_k$ at any other bank $k \neq i$ that holds similar assets.[22] We assume that indirect losses stemming from a fire sale are insufficient to ignite defaults at banks not hit by direct asset losses. Technically, this assumption amounts to the parametric constraint,

$$\mu > \frac{\lambda}{(1-\mu)(1-\rho)}, \tag{18}$$

Intuitively, this assumption posits the dominance of reputation cost over the fire-sale cost. (See also footnotes 36 and 37.) Indeed, this constraint assures that banks in the financial network do not walk away from their sponsored debt obligations as result of indirect (fire-sale) losses to collateral assets. Consequently, all banks $k \neq i$ indeed choose to meet the fire-sale induced shortfalls λc_k by liquidating a portion η_{xk} of their own illiquid assets,

$$\eta_{xk} = \frac{\lambda c_k}{(1-\rho)e_k}, \qquad \forall k \in \mathcal{N} \setminus \{i\}, \tag{19}$$

in accordance with protocol (17). The above parametric assumption (18) limits losses from a fire sale in the model. This ultimately generates a conservative (lower-bound) estimate of the differential effect of sequential losses (versus a single loss) as we see in later sections.

[22] For simplicity, we assume the same fire-sale cost λ whether one bank i or several such banks originally walk away from their sponsored debt obligations, thus prompting their creditors to seize and sell the associated collateral.

Clearing Payment Mechanism

Let $\boldsymbol{e}(\boldsymbol{x}) \in (\boldsymbol{R}^{+})^n$ denote the vector of ex-post (book) values of the illiquid assets of banks after the loss vector $\boldsymbol{x}$ hits the financial system. For a bank i that is hit directly by these losses ($x_i > 0$), we have,

$$e_i(\boldsymbol{x}) \;=\; \begin{cases} (1-\mu)e_i, & \text{if} \quad \frac{x_i}{1-\rho} \geq \mu e_i \quad \text{(walk-away)}, \\ \\ e_i - \frac{x_i}{1-\rho} & \text{if} \quad \frac{x_i}{1-\rho} < \mu e_i \quad \text{(stay)}. \end{cases} \tag{20}$$

If bank k is not hit directly by these losses ($x_k = 0$), k stays with its sponsored debt obligations (19), and after liquidation its illiquid asset value reduces to,

$$e_k(\boldsymbol{x}) = e_k - \eta_{xk} e_k = e_k - \frac{\lambda c_k}{1-\rho}.$$

Similar to E-N setting (5), following decisions and actions of all banks associated with external losses $\boldsymbol{x}$, the financial system reaches a clearing state characterized by a clearing payment vector $\boldsymbol{p}(\boldsymbol{x})$. The latter is a fixed point $\boldsymbol{p}$ of the following map,

$$\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{e}, \boldsymbol{b}, \boldsymbol{x}) = \min\left\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{e}(\boldsymbol{x})\ ,\ \overline{\boldsymbol{p}}\right\}, \tag{21}$$

where $\boldsymbol{e}(\boldsymbol{x})$ is defined in (20). Three intuitions underlying this map are as follows.

First, each bank i, which suffers a direct loss x_i to its sponsored assets c_i, makes a contingent decision (to either stay with or walk away from sponsored debt obligations) based on its self-centered valuation (17). The bank's illiquid asset value $\boldsymbol{e}(\boldsymbol{x})$ follows (20), which then implies that the bank's total interbank asset value (i.e., intra-network but net of sponsored debts) is contingent on whether it walks away or stays,

$$\sum_{j=1}^{n} a_{ji} p_j(\boldsymbol{x}) + e_i(\boldsymbol{x}) + c_i(\boldsymbol{x}) - d_i(\boldsymbol{x}) \;=\; \begin{cases} \sum_{j=1}^{n} a_{ji} p_j(\boldsymbol{x}) + (1-\mu)e_i & \text{if} \quad \frac{x_i}{1-\rho} \geq \mu e_i \text{ (walk-away)}, \\ \\ \sum_{j=1}^{n} a_{ji} p_j(\boldsymbol{x}) + e_i - \frac{x_i}{1-\rho} & \text{if} \quad \frac{x_i}{1-\rho} < \mu e_i \text{ (stay)}. \end{cases} \tag{22}$$

Thus, once the decision at every bank is made based on its own self-centered perspective, the accounts with sponsored creditors are settled first (settlement priority). We are then back to the E-N clearing mechanism, in which the clearing payment vector $\boldsymbol{e}(\boldsymbol{x})$ is determined as markets clear.

Second, in case a bank i chooses to walk away from its sponsored debt obligations, i's asset side of the balance sheet records a total asset value of $\sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + (1-\mu)e_i$. The liability side records a total debt of $\overline{p}_i$. This is because once i walks away, sponsored creditors $\mathcal{I}$ seize assets c_i, and at the same time i is relieved from debts d_i. In accordance with the E-N default mapping (Definition 2), if $\sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x})+(1-\mu)e_i > \overline{p}_i$, then bank i is not in default to its interbank obligations (i.e., $p_{ij} > 0$), and not to its other outside debts (i.e., $b_i > 0$). As a result, $p_i(\boldsymbol{x}) = \overline{p}_i$. On the other hand, if $\sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + (1-\mu)e_i < \overline{p}_i$, bank i defaults and transfers its ex-post assets to interbank and other outside creditors on a pro-rata basis. Bank i's liability then is written down to $p_i(\boldsymbol{x}) = \sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + (1-\mu)e_i$.

Third, if bank i chooses to stay, it first posts extra collateral to meet its margin call, and subsequently i's illiquid assets drop to $\sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + e_i - \frac{x_i}{1-\rho}$. Then network markets clear in accordance with E-N mapping (Definition 2). If $\sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + e_i - \frac{x_i}{1-\rho} > \overline{p}_i$, then i is not in default, and $p_i(\boldsymbol{x}) = \overline{p}_i$. Otherwise, if $\sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + e_i - \frac{x_i}{1-\rho} < \overline{p}_i$ then bank i is in default eventually, and the network obligation vector $\boldsymbol{p}(\boldsymbol{x})$ solves $p_i(\boldsymbol{x}) = \sum_{j=1}^{n} a_{ji}p_j(\boldsymbol{x}) + e_i - \frac{x_i}{1-\rho}$.

All in all, the resulting clearing payment vector in every case is the fixed point of the map Φ in (21). In retrospect, due to the complexity of the financial network, banks are not in a position to reliably and strategically coordinate the ultimate clearing payment vector. In the circumstance of having just experienced losses $\boldsymbol{x}$ and under time pressure to act, it is plausible that banks make decisions (to either stay with or walk away from their sponsored debt obligations) based on their self-centered valuation protocol (17). The following results demonstrate that this self-centered decision making ("revealed preference") is consistent (their proof is relegated to Appendix B.1).

Proposition 6 *After losses $\boldsymbol{x}$ hit the financial system, assume that every bank i makes a decision regarding its implicit guarantees to its sponsored creditors $\mathcal{I}$ in accordance with its self-centered decision-making protocol (17).*

(i) If bank i walks away, its sponsored creditors $\mathcal{I}$ will suffer a net loss bounded from below, and i surely will not be in outright default after markets clear.

(ii) If bank i stays, its sponsored creditors $\mathcal{I}$ will recover their full loans, and i's remaining assets have strictly higher value than in the case where it walks way. Combined with result (i), it is implied that i also surely will not be in outright default after markets clear.

(iii) The combination of the two results above implies that, in all scenarios, no bank in the financial network is ever in outright default after markets clear.

This proposition clearly shows that outright defaults are eliminated if banks in the network act in a self-centered manner, in stark contrast with the original E-N setting (which does not feature self-centered decision making by banks). This result reflects an important risk that sponsored creditors face in reality in bad times. This risk is associated with the non-binding (implicit) guarantees made by financial institutions. Even if banks deal with these sponsored creditors before other (interbank and outside) creditors, this "advantage" can be lost when banks back away from their implicit guarantees under either extreme losses, mild reputation damages, or their combinations.

Existence and Uniqueness of the Clearing Payment Vector

From the previous discussion, when bank i is hit with a loss x_i, its sponsored creditors $\mathcal{I}$ get

$$d_i(\boldsymbol{x}) \; = \begin{cases} c_i - x_i \quad \text{if} \quad \frac{x_i}{1-\rho} \geq \mu e_i \quad (i \text{ walks away}), \\ \\ c_i \qquad \text{if} \quad \frac{x_i}{1-\rho} < \mu e_i \quad (i \text{ stays}), \end{cases} \qquad \forall i \in \mathcal{N}. \tag{23}$$

The vector of payments $\boldsymbol{p}(\boldsymbol{x})$ that are due to interbank counter-parties and outside creditors is a fixed point of the map Φ in (21). Using notation from Section 2.1, the ex-post financial system can be characterized by the tuple $\{\boldsymbol{A}, \bar{\boldsymbol{p}}, \boldsymbol{e}(\boldsymbol{x}), \boldsymbol{b}\}$. Since this system has the same relative liability matrix $\boldsymbol{A}$, and the same vector of outside liabilities as the original E-N financial system, the sufficient condition underlying Proposition 1 remains intact. Consequently, the map (21) has a unique fixed point. We summarize this result in Proposition 7 below.

Proposition 7 *Let losses $\boldsymbol{x}$ hit the financial system $\{\boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{d}\}$. Then,*

(i) payments $\boldsymbol{d}(\boldsymbol{x})$ to sponsored creditors $\mathcal{I}$ are uniquely determined by (23),

(ii) payments $\boldsymbol{p}(\boldsymbol{x})$ to interbank and other creditors are given by the unique fixed point of the map (21),

in the clearing markets.

These results simply indicate that the uniqueness of market-clearing solution, the salient property of the original E-N financial network, is maintained in our extended setting.

4 Sequential Losses Hitting a Single Bank

In this section we study and demonstrate the relevance of a loss sequence hitting a single bank in the generalized framework of the financial system of Section 3. To model losses hitting a particular bank i, we consider two loss vectors,

$$\boldsymbol{x} = (0, \ldots, 0\ , x_i = x,\ 0, \ldots, 0)^T, \qquad \boldsymbol{y} = (0, \ldots, 0\ , y_i = y,\ 0, \ldots, 0)^T.$$

that only directly impact bank i. We first analyze the loss sequence $\boldsymbol{x}$ then $\boldsymbol{y}$, denoted hereafter by $\boldsymbol{x}\to\boldsymbol{y}$. When bank i is hit first with a loss of sufficiently small magnitude (17), $x \leq \mu(1-\rho)e_i$, its reputation loss outweighs its combined direct loss and liquidation cost. Consequently, for such moderate loss x, bank i opts to post additional collateral to the clearing house and stays with its sponsored debt obligations. Otherwise, if $x > \mu(1-\rho)e_i$, i walks away after the initial loss x.

To make the analysis specific, first assume that bank i indeed has chosen to stay after the initial moderate loss x. As a result, i now has "less skin in the game" (thus, lower reputation concern) because its illiquid assets have been reduced to $e_i(x) = e_i - \frac{x}{1-\rho}$ (20). As the second loss y hits, if i walks away, it incurs losses from reputation damage, $\mu e_i(x)$. If i stays, it incurs direct and liquidation losses whose combined value is $\frac{y}{1-\rho}$. When the second loss y is such that,

$$\mu e_i(x) > \frac{y}{1-\rho}, \qquad \text{where} \qquad e_i(x) = e_i - \frac{x}{1-\rho}, \tag{24}$$

bank i chooses to stay with its sponsored debt obligations following sequential losses $\boldsymbol{x}\to\boldsymbol{y}$. Otherwise, i walks away after the second loss y, similar to (17). To put relative magnitudes of sequential losses into perspective, we rewrite the above threshold (for bank i to also stay after the second loss) in terms of the cumulative loss M,[23]

$$x \geq \frac{M - (1-\rho)\mu e_i}{1-\mu}, \qquad M \equiv x + y. \tag{25}$$

Intuitively, given a cumulative loss M of appropriate magnitude (specified in (26) below), when the first loss x is sufficiently large,[24] the second loss y is sufficiently small to prompt i to stay after this second loss y. We limit ourselves to a situation in which the cumulative loss to bank i being

[23] The threshold is conditional on i having stayed after the first loss x, and is obtained under (14).

[24] Yet, in the current scenario (for illustration), x is also presumably sufficiently moderate that bank i stays with its sponsored debt obligations after this initial loss x. We analyze all remaining scenarios next.

bounded from above by,

$$M \leq \mu(2-\mu)(1-\rho)e_i. \tag{26}$$

The parametric region defined by this bound is rich; (26) assures that the threshold (17) for i to stay after initial loss x is compatible with the threshold (25) for i to also stay after the second loss y. Otherwise, if the cumulative loss M exceeds the bound (26), bank i surely walks away after sequential losses $\boldsymbol{x} \to \boldsymbol{y}$ regardless of how the total loss M is partitioned into x and y, which is a less interesting context.[25] Finally, under constraint (18), all other banks $k \neq i$ (i.e., those do not suffer from direct asset losses) opt to stay with their sponsored debt obligations.

4.1 Sequential Losses

We now comprehensively discuss all possible strategic responses of bank i to the sequence of (unexpected) losses $\boldsymbol{x} \to \boldsymbol{y}$. The clearing payment vector resulting from this loss sequence is the unique fixed point of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \bar{\boldsymbol{p}}, \boldsymbol{e}, \boldsymbol{b}, \boldsymbol{x}\to\boldsymbol{y})$ defined in (21). There are three mutually exclusive and exhaustive cases.

Case 1: $0 \leq x < \frac{M-(1-\rho)\mu e_i}{1-\mu}$. Following an initial small loss x in this range, bank i chooses to meet the margin call.[26] But i also opts to default after the second (unexpected) loss y hits the bank, because under assumption (26), $\frac{y}{1-\rho} > \mu\left(e_i - \frac{x}{1-\rho}\right)$ for such a moderate initial loss x. In this case, bank i's total asset value reduces to $e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)(e_i - \frac{x}{1-\rho})$ due to the liquidation (after x hits) and reputation losses (after y hits). As bank i fails to post the second collateral shortfall, its collateral is seized and sold off by creditors causing the value of collateral at other banks to drop by the factor $\lambda \in (0,1)$. Each bank $j \neq i$ then receives a margin call in the respective amount of λc_j and chooses to liquidate a portion $\eta_j = \frac{\lambda c_j}{(1-\rho)e_j}$ (19) of their illiquid assets e_j.[27] The total loss to a bank $j \neq i$ therefore is equal to $\lambda c_j + \rho\eta_j e_j = \lambda c_j + \frac{\rho}{1-\rho}\lambda c_j = \frac{\lambda c_j}{1-\rho}$. As a result, banks' illiquid asset values are as follows,

$$e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)\left(e_i - \frac{x}{1-\rho}\right), \qquad e_j(\boldsymbol{x}\to\boldsymbol{y}) = e_j - \frac{\lambda c_j}{1-\rho} = \left(1 - \frac{\lambda}{1-\rho}\right)e_j, \quad \forall j \in \mathcal{N}\setminus i. \tag{27}$$

[25] To see this, suppose M is large and (26) reverses. If the initial loss x is moderate (17) and i stays, then the second loss y is sizable (25) and i walks away right after y hits. Alternatively, the first loss x is sizable, and i just walks away right after x hits.

[26] To see this, substituting (26) into this bound implies, $0 \leq x < \mu(1-\rho)e_i$. Therefore, bank i stays after loss x in accordance with (17).

[27] Recall that by virtue of the parametric assumption (18), secondary banks $j \neq i$, whose sponsored assets did not suffer directly from external losses ($x_k = 0$), do not default because of fire-sale devaluation.

The total loss $L(\boldsymbol{x}\to\boldsymbol{y})$ to the financial system, and loss $l_i(\boldsymbol{x}\to\boldsymbol{y})$ to bank i's interbank creditors (which takes into account only the debt write-offs due to possible defaults of member banks) respectively are,

$$
\begin{aligned}
L(\boldsymbol{x}\to\boldsymbol{y}) &= \frac{x}{1-\rho} + \mu\left(e_i - \frac{x}{1-\rho}\right) + y + S(\boldsymbol{x}\to\boldsymbol{y}) + \sum_{j\neq i} \frac{\lambda c_j}{1-\rho}, &(28)\\
l_i(\boldsymbol{x}\to\boldsymbol{y}) &= \overline{p}_i - p_i(\boldsymbol{x}\to\boldsymbol{y}) + \sum_{\{j:\ \overline{p}_{ij}>0\}} \frac{\lambda c_j}{1-\rho}. &(29)
\end{aligned}
$$

In (28), $\frac{x}{1-\rho}$ accounts for the combined loss from the first direct loss x and the associated liquidation, $\mu\left(e_i - \frac{x}{1-\rho}\right)$ for the reputation damage to bank i as it walks away upon the second loss y.[28] Also in (28), the next term y accounts for the loss to i's creditors as a result of i's default, and $\frac{\lambda c_j}{1-\rho}$ for the loss to a bank $j \neq i$ due to the fire-sale negative externality. Finally, following Glasserman and Young (2015), $S(\boldsymbol{x}\to\boldsymbol{y})$ denotes the total shortfall in payments of the financial system as a result of the sequential losses $\boldsymbol{x}\to\boldsymbol{y}$,

$$
S(\boldsymbol{x}\to\boldsymbol{y}) \equiv \sum_{i\in\mathcal{N}} \overline{p}_i - p_i(\boldsymbol{x}\to\boldsymbol{y}), \tag{30}
$$

a quantity that is determined as markets clear.

Case 2: $\frac{M-(1-\rho)\mu e_i}{1-\mu} \leq x \leq (1-\rho)\mu e_i$. A moderate initial loss x in this range also indicates a second moderate loss y by virtue of the cumulative loss M satisfying (26). In this case, bank i chooses to meet the margin calls after every loss in the sequence $\boldsymbol{x}\to\boldsymbol{y}$. As a result, i's illiquid asset value drops by $\frac{x+y}{1-\rho}$. Since there is no fire sale, asset values at all other banks remain unchanged. Thus, the illiquid asset values of all banks are as follows,

$$
e_i(\boldsymbol{x}\to\boldsymbol{y}) = e_i - \frac{x+y}{1-\rho}, \qquad e_j(\boldsymbol{x}\to\boldsymbol{y}) = e_j, \quad \forall j \in \mathcal{N}\setminus i. \tag{31}
$$

Similarly, losses respectively to the system and to bank i's creditors are,

$$
L(\boldsymbol{x}\to\boldsymbol{y}) = \frac{x+y}{1-\rho} + S(\boldsymbol{x}\to\boldsymbol{y}), \qquad l_i(\boldsymbol{x}\to\boldsymbol{y}) = \overline{p}_i - p_i(\boldsymbol{x}\to\boldsymbol{y}), \tag{32}
$$

which counts only debt write-offs due to possible defaults of network banks, and $S(\boldsymbol{x}\to\boldsymbol{y})$ denotes total shortfalls in payments (30) of the current case.

[28] At which point, i's "skin in the game," or illiquid asset value, has been reduced to $\left(e_i - \frac{x}{1-\rho}\right)$ due to the fire sale associated with the first loss x.

In comparison with case 1 above, in case 2 the cumulative loss M is partitioned into a sequence of two moderate losses x, y, in such a way that bank i chooses not to walk away from its sponsored debt obligations. Consequently, there is no fire sale, and no losses from a fire sale arise in the network. The absence of fire-sale negative externalities to the financial system in a sequence of moderate losses is an important feature because of its relevance for the public bailout policy of failed banks as we discuss below.

Case 3: $x > (1-\rho)\mu e_i$. Following a large initial loss x of this magnitude, bank i opts to walk away immediately. In this case, bank i suffers a reputation loss of μe_i and each bank $j \neq i$ incurs a loss of $\frac{\lambda c_j}{1-\rho}$ due to a fire sale of bank i's collateral. Banks' asset values, and losses to the system and to bank i's creditors respectively are,[29]

$$e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)e_i, \qquad e_j(\boldsymbol{x}\to\boldsymbol{y}) = \left(1 - \frac{\lambda}{1-\rho}\right)e_j, \quad \forall j \in \mathcal{N}\setminus i, \tag{33}$$

$$L(\boldsymbol{x}\to\boldsymbol{y}) = x+y+\mu e_i+S(\boldsymbol{x})+\sum_{j\neq i}\frac{\lambda c_j}{1-\rho}, \qquad l_i(\boldsymbol{x}\to\boldsymbol{y}) = x+(\overline{p}_i-p_i(\boldsymbol{x}\to\boldsymbol{y}))+\sum_{\{j:\ \overline{p}_{ij}>0\}}\frac{\lambda c_j}{1-\rho}. \tag{34}$$

We next turn to the scenario in which a single loss of magnitude $M = x + y$ hits bank i before presenting a comparative analysis of sequential versus single loss configurations.

4.2 Single Loss of Equal Cumulative Magnitude

After a single loss $M = x + y$ hitting bank i, if i decides to stay, it needs to liquidate a portion $\eta = \frac{M}{(1-\rho)e_i}$ of its illiquid asset e_i while realizing a combined loss of $\frac{M}{1-\rho}$ (of direct loss M and liquidation cost $\rho\eta e_i$). When M is sufficiently large, or $M \geq (1-\rho)\mu e_i$, the above combined loss to bank i dominates its reputation loss.[30] Accordingly, bank i opts to walk away. The resulting fire sale negatively affects other banks' collateral and prompts margin calls and further liquidations of illiquid assets. Thus, the illiquid asset values (after losses) of all banks are as follows,

$$e_i(\boldsymbol{x}+\boldsymbol{y}) = (1-\mu)e_i, \qquad e_j(\boldsymbol{x}+\boldsymbol{y}) = \left(1 - \frac{\lambda}{1-\rho}\right)e_j, \quad \forall j \in \mathcal{N}\setminus i. \tag{35}$$

[29] For simplicity, we assume that i's collateral is sold to an entity outside the financial system (e.g., government), which fully internalizes the second loss y. However, losses in this stylized model give us lower (conservative) estimates for the losses in the more realistic situation in which i's collateral is sold to banks in the financial network.

[30] Note that this lower bound on the single loss does not negate its upper bound (26) since $\mu \in [0,1]$ implies $(1-\rho)\mu e_i \leq \mu(2-\mu)(1-\rho)e_i$.

The clearing vector associated with the loss M is the unique fixed point of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{e}, \boldsymbol{b}, \boldsymbol{x}{+}\boldsymbol{y})$. The total losses to the financial system and to creditors of bank i respectively are,

$$L(\boldsymbol{x}{+}\boldsymbol{y}) = x + y + \mu e_i + S(\boldsymbol{x}{+}\boldsymbol{y}) + \sum_{j \neq i} \frac{\lambda c_j}{1-\rho}, \qquad l_i(\boldsymbol{x}{+}\boldsymbol{y}) = x + y + (\overline{p}_i - p_i(\boldsymbol{x}{+}\boldsymbol{y})) + \sum_{\{j:\, \overline{p}_{ij} > 0\}} \frac{\lambda c_j}{1-\rho}. \tag{36}$$

Here $S(\boldsymbol{x}{+}\boldsymbol{y}) \equiv \sum_{i \in \mathcal{N}} \overline{p}_i - p_i(\boldsymbol{x}{+}\boldsymbol{y})$ denotes the total shortfall in payments (the counterpart of (30) for the current scenario).

Discussion

We observe that banks' illiquid assets (27), (31), (33) in all cases of the sequential-loss scenario differ from those in (35) of the single-loss scenario in general, $\boldsymbol{e}(\boldsymbol{x}{\to}\boldsymbol{y}) \neq \boldsymbol{e}(\boldsymbol{x}{+}\boldsymbol{y})$. Consequently, the associated clearing vectors also differ in these two scenarios. This implies that the impacts on the financial system produced by a sequence of (unexpected) losses $\boldsymbol{x}{\to}\boldsymbol{y}$ and alternatively by the single loss $\boldsymbol{x}{+}\boldsymbol{y}$ of the same cumulative magnitude can be very different. More colloquially, an earthquake produces different results than a cascade of dominoes.[31] The difference arises from the features introduced in Section 3, which include self-centered (myopic and strategic) decisions by banks hit with direct losses, the settlement priority to sponsored creditors, and the fire-sale devaluation of similar illiquid assets held by banks. When every loss in the sequence is moderate, bank i (hit directly by these losses) finds it beneficial to stay with its sponsored debt obligations, and thus spares the financial system from a costly fire sale. Accordingly, as an implication, the government may find it welfare-improving to intervene and partition a single realized loss $M = x{+}y$ to bank i into a sequence of moderate losses $\boldsymbol{x}{\to}\boldsymbol{y}$. This can be done, e.g., by lending the amount of y to bank i immediately after the loss M hits (effectively reducing original loss M to an initial loss x to bank i) and then recovering this bailout loan y from i after markets have stabilized from the initial loss x.

A caveat of our comparative statics would be that in the case a single loss M is engineered into a sequence $\boldsymbol{x}{\to}\boldsymbol{y}$, the second loss y is considered unexpected in the analysis of Section 4.1. Whereas, bank i should be fully aware (already at the time M hits) that it has to pay back loan y to the government in the future, i.e., the second loss y is hardly unexpected. However, as we illustrate in

[31] More precisely, our current comparison concerns a cascade of unsuspecting dominoes because the shocks in the sequential-loss configuration are currently considered uncoordinated.

a numerical example below, when the single realized loss $M = x + y$ to bank i is sufficiently large (yet preserving bound (26)), i's equity holders lose everything regardless of how one partitions M into sequential losses $\boldsymbol{x}{\to}\boldsymbol{y}$ (Figure 5). In this situation, the moral hazard issues (the perspective of i's equity holders) do not arise. It is then reasonable that i's equity holders, being indifferent to loss sequencing, would act on a good-faith basis to minimize the losses to i's creditors. But as these creditors clearly benefit from sequencing the original loss M into moderate losses $\boldsymbol{x}{\to}\boldsymbol{y}$ (Figure 5) to prevent a fire sale, bank i's full expectation of the incoming second loss y in the sequence should not prevent (from the view of bank i's incentive compatibility constraint) the government's desire to intervene and break M into $\boldsymbol{x}{\to}\boldsymbol{y}$. Put differently, at least for such values of the single realized loss M, our analysis indicates the relevance of loss sequencing in the financial system.

4.3 Numerical Illustration

To illustrate the analysis presented in Sections 4.1-4.2, we consider a stylized numerical example of a financial network of three banks, whose (interbank and outside) assets and liabilities are shown in Figure 1. Consider two losses $\boldsymbol{x}, \boldsymbol{y}$ that hit bank 1; $\boldsymbol{x} = (x, 0, 0)$, $\boldsymbol{y} = (y, 0, 0)$ with aggregate loss

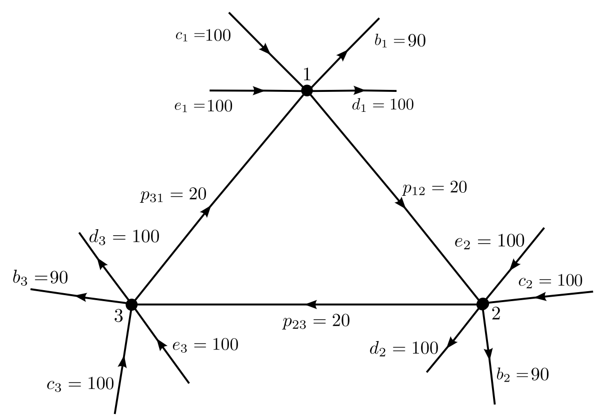

Figure 1: A network with three symmetric banks. Incoming arrows to and outgoing arrows from a bank respectively depict assets and liabilities of that bank. The liquidation, reputation and fire-sale parameters respectively are $\rho = 0.2, \mu = 0.4, \lambda = 0.3$, and aggregate loss $M = 40$.

$M \equiv x + y = 40$.

We first consider the single loss $\boldsymbol{x}{+}\boldsymbol{y}$. Note that bank 1 will choose to default since $x + y = 40 > 32 = (1 - \rho)\mu e_i$. From Section 4, the clearing vector resulting from the single loss $\boldsymbol{x}{+}\boldsymbol{y}$ is the unique fixed point of the map,

$$\Phi(\boldsymbol{p}) = \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y}), \overline{\boldsymbol{p}}\}. \tag{37}$$

where $\boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y}) = ((1-\mu)e_1, e_2 - \frac{\lambda c_2}{1-\rho}, e_3 - \frac{\lambda c_3}{1-\rho})$. Here the relative liability matrix is,

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{2}{11} & 0 \\ 0 & 0 & \frac{2}{11} \\ \frac{2}{11} & 0 & 0 \end{bmatrix}.$$

Total Shortfall in Payments by Banks

Figure 2 below shows the total shortfall in payments by banks for various scenarios (computed at the respective market clearings of these scenarios). In particular, we observe a sizable reduction in

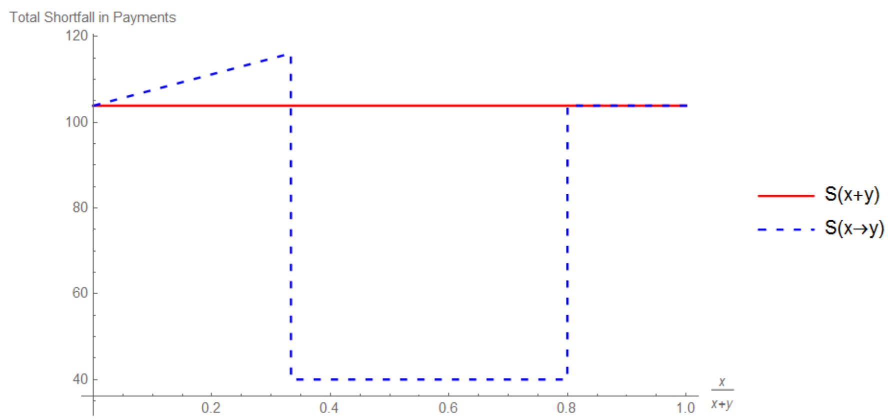

Figure 2: Total payment shortfalls in the financial system: (i) under sequential losses $S(\boldsymbol{x}\to\boldsymbol{y})$ (in (28), (32), or (34) for appropriate value of losses x, y), and (ii) under a single loss $S(\boldsymbol{x}+\boldsymbol{y})$ (in (36)). Losses hit bank 1 in a network as depicted in Figure 1.

total payment shortfalls $S(\boldsymbol{x}\to\boldsymbol{y})$ in the sequential-loss scenario $\boldsymbol{x}\to\boldsymbol{y}$ for moderate values of losses x, y. The intuition is as follows. If the first loss x is small relative to the sum of the losses $M = x+y$, bank 1 chooses to stay. This means that the bank must liquidate a portion of its illiquid asset e_1 and incurs a liquidation loss of $\frac{x}{1-\rho}$. When the bank is hit by a second loss y in the sequence, which is large relative to $M = x + y$, it will then choose to walk away, which will impose losses at other banks through the fire-sale effect. Thus, the clearing payment vector $\boldsymbol{p}(\boldsymbol{x}\to\boldsymbol{y})$ is the fixed point of the map $\Phi(\boldsymbol{p}) = \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{e}(\boldsymbol{x}\to\boldsymbol{y}), \overline{\boldsymbol{p}}\}$, where $\boldsymbol{e}(\boldsymbol{x}\to\boldsymbol{y}) = ((1-\mu)(e_1 - \frac{x}{1-\rho}), e_2 - \frac{\lambda c_2}{1-\rho}, e_3 - \frac{\lambda c_3}{1-\rho})$.

In contrast, if bank 1 is hit by a single loss of magnitude $M = x + y$, it will choose to walk away immediately and avoid the liquidation loss and the clearing payment vector $\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y})$ is the fixed point of the map Φ in (37). It is obvious that $\boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y}) > \boldsymbol{e}(\boldsymbol{x}\rightarrow\boldsymbol{y})$ and as a consequence, the clearing payment vector $\boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y})$ is always greater than its counterpart $\boldsymbol{p}(\boldsymbol{x}\rightarrow\boldsymbol{y})$. This implies that for relatively small x, $S(\boldsymbol{x}\rightarrow\boldsymbol{y}) > S(\boldsymbol{x}+\boldsymbol{y})$. Figure 2 clearly reflects this pattern.

When the two losses x, y in the sequence are of similar size, it is optimal for bank 1 to post more collateral after each loss. By internalizing its losses, bank 1 spares the financial system from a fire sale that would adversely affect many other banks: $S(\boldsymbol{x}\rightarrow\boldsymbol{y}) \ll S(\boldsymbol{x}+\boldsymbol{y})$. As evident in Figure 2, the total losses in the two loss scenarios differ starkly.

If the first loss x is large relative to the sum of the two losses $M = x + y$, bank 1 will walk away immediately. We assume that the bank's creditors will seize and sell off its collateral to an entity outside the financial system. As such, when the second loss y hits, the financial system is not affected. Furthermore, bank 1 will only incur reputation damage, and will avoid any liquidation loss in this case. As a result, the balance sheet values at banks are the same in both cases: $\boldsymbol{e}(\boldsymbol{x}\rightarrow\boldsymbol{y}) = ((1-\mu)e_1, e_2 - \frac{\lambda c_2}{1-\rho}, e_3 - \frac{\lambda c_3}{1-\rho}) = \boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y})$. Accordingly, banks face the same clearing payments and hence, the same shortfalls under the two scenarios of losses: $S(\boldsymbol{x}\rightarrow\boldsymbol{y}) = S(\boldsymbol{x}+\boldsymbol{y})$.

Total Loss to the Financial System

The total loss to the financial system exhibits similar patterns as the total shortfall in payments by banks. On one hand, if bank 1 is hit by a single loss of magnitude $M = x + y$, the total loss to the financial system is, $L(\boldsymbol{x}+\boldsymbol{y}) = x + y + \mu e_i + S(\boldsymbol{x}+\boldsymbol{y}) + \sum_{j\neq i} \frac{\lambda c_j}{1-\rho}$. On the other hand, when bank 1 is hit by a sequence of losses $\boldsymbol{x}\rightarrow\boldsymbol{y}$ where the first loss x is small relative to $M = x + y$, we have

$$\begin{aligned} L(\boldsymbol{x}\rightarrow\boldsymbol{y}) &= \frac{x}{1-\rho} + \mu(e_1 - \frac{x}{1-\rho}) + y + S(\boldsymbol{x}\rightarrow\boldsymbol{y}) + \sum_{j\neq i} \frac{\lambda c_j}{1-\rho} \\ &= \frac{1-\mu}{1-\rho}x + y + \mu e_1 + S(\boldsymbol{x}\rightarrow\boldsymbol{y}) + \sum_{j\neq i} \frac{\lambda c_j}{1-\rho}. \end{aligned} \tag{38}$$

We already showed in the discussion above that $S(\boldsymbol{x}\rightarrow\boldsymbol{y}) > S(\boldsymbol{x}+\boldsymbol{y})$ for x in this range. If in addition the reputation-damage cost μ is smaller than the liquidation cost ρ, then total loss to the financial system is smaller when shocks hit sequentially, $L(\boldsymbol{x}\rightarrow\boldsymbol{y}) > L(\boldsymbol{x}+\boldsymbol{y})$. Figure 3, which plots the total losses to the financial system under the two scenarios, illustrates this assertion. When the sizes of the two losses x, y are relatively close to each other, bank 1 will stay after it is hit

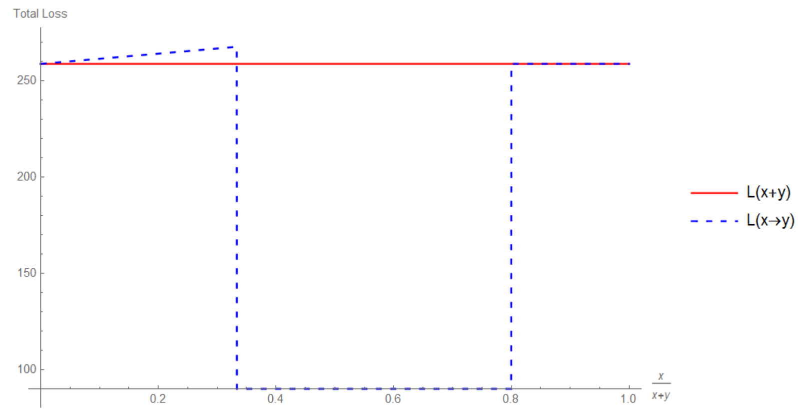

Figure 3: Total losses of the financial system: (i) under sequential losses $L(\boldsymbol{x}\to\boldsymbol{y})$ (given by (28), (32), or (34) for appropriate value of losses x, y), and (ii) under a single loss $L(\boldsymbol{x}+\boldsymbol{y})$ (given by (36)). Losses hit bank 1 in a network as depicted in Figure 1.

by these losses, and the total loss to the financial system is, $L(\boldsymbol{x}\to\boldsymbol{y}) = \frac{x+y}{1-\rho} + S(\boldsymbol{x}\to\boldsymbol{y})$. Since $S(\boldsymbol{x}\to\boldsymbol{y}) \ll S(\boldsymbol{x}+\boldsymbol{y})$ in this region, we also have that $L(\boldsymbol{x}\to\boldsymbol{y}) \ll L(\boldsymbol{x}+\boldsymbol{y})$ as shown in Figure 3.

Lastly, when the first loss x is large relative to the sum of the two losses $M = x + y$, bank 1 will walk away immediately. Consequently, the total loss is,[32] $L(\boldsymbol{x}\to\boldsymbol{y}) = x + y + \mu e_i + S(\boldsymbol{x}\to\boldsymbol{y})$. Observe in this case that $S(\boldsymbol{x}\to\boldsymbol{y}) = S(\boldsymbol{x})$ since the second loss y has no impact on the financial system. We have shown above that $S(\boldsymbol{x}\to\boldsymbol{y}) = S(\boldsymbol{x}+\boldsymbol{y})$ when x and y are sufficiently close in size. This means that we also have $L(\boldsymbol{x}\to\boldsymbol{y}) = L(\boldsymbol{x}+\boldsymbol{y})$, see Figure 3.

Loss to Equity Holders

Banks make decisions to minimize losses to their equity values. We distinguish between myopic losses and eventual losses to banks' equity holders. Here myopic losses refer to losses to banks when they disregard the externalities that their actions impose on the network, whereas eventual losses are the realized losses after taking into account network (market-clearing) effects. Given a realistically complex network, banks are able to ascertain myopic losses but most likely are not in a position to decipher eventual losses before markets clear. When banks respond to shocks based on their myopic (i.e., self-centered) loss calculations, but their responses also result in eventual losses that are consistent with those myopic responses, then banks' self-centered actions are self-fulfilling and consistent.

[32] We can think of this as a lower bound of the total loss to the financial system. If the collateral is bought by another bank in the system, the second loss will have an adverse affect and likely cause further losses.

In our current numerical example, it turns out that the myopic and eventual losses are identical; therefore banks' self-centered decisions are consistent.[33] The myopic losses to bank 1's equity holders in term of original losses $\boldsymbol{x}$ are determined as follows,

$$\begin{array}{c}\text{bank 1's myopic loss} \\ \text{to equity holders}\end{array} = \begin{cases} \min\{\overline{w}_1, \frac{x}{1-\rho}\} & \text{if} \quad x < \mu(1-\rho)e_1 \text{ (stay)} \\ \min\{\overline{w}_1, (1-\mu)e_1\} & \text{if} \quad x \geq \mu(1-\rho)e_1 \text{ (walk-away)}, \end{cases} \tag{39}$$

$$\text{where} \quad \overline{w}_1 \equiv (\boldsymbol{A}^T\overline{\boldsymbol{p}})_1 + c_1 + e_1 - b_1 - d_1 - \overline{p}_1 = (\boldsymbol{A}^T\overline{\boldsymbol{p}})_1 + c_1 - b_1 - \overline{p}_1.$$

Here $\overline{w}_1$ is the ex-ante equity value (or ex-ante net worth) of bank 1. In the current numerical example depicted in Figure 1, $\overline{w}_1 = 10$. To illustrate the self-centered decision making of a bank after a direct loss, Figure 4 plots the losses to bank 1's equity holders and bank 1's network creditors resulting from the first loss x.[34] When the first loss is small, $0 < x < (1-\rho)\overline{w}_1$ (39), bank 1's

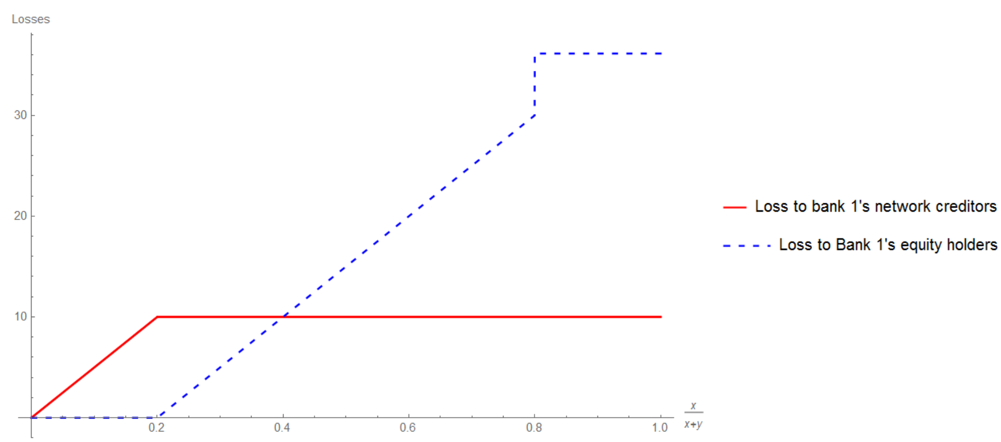

Figure 4: Total loss of bank 1's equity holders (39), and total loss of bank 1's network creditors (i.e., banks 2 and 3) resulting from (only) the first loss x. The associated banking network is depicted in Figure 1. Bank 1's equity is cleared out when $\frac{x}{1-\rho} = 10$ or $x = 8$, which is smaller than the threshold of stay/walk-away for the bank, which is $(1-\rho)\mu e_1 = 32$.

equity-holder loss $\frac{x}{1-\rho}$ increases linearly with x. For a larger loss, $x > (1-\rho)\overline{w}_1$, bank 1's equity-holder loss stabilizes at w_1 (they lose everything, or the equity value drops to zero). Bank 1's network creditors lose nothing when the loss is small, $0 < x < (1-\rho)\overline{w}_1$, because in this case bank 1 opts to stay with their sponsored debt obligations, and bank 1 has sufficient resources to meet margin calls fully (equity value is strictly positive after the loss). Hence, the loss x does not spill

[33]In particular, bank 1's equity-holder loss from its self-centered perspective is identical to bank 1 equity holders' loss realized after markets clear (following loss x).

[34]In the numerical example depicted in Figure 1, network creditors of bank 1 are banks 2 and 3. These banks are not sponsored creditors of bank 1 in this specific example.

over to the financial network. For a moderate loss, $(1-\rho)\overline{w}_1 < x < \mu(1-\rho)e_1$, bank 1 opts to stay with its sponsored debt obligations, but has insufficient resources to meet margin calls fully (equity value is zero after the loss). Therefore, in this range bank 1 is insolvent, and the loss x leads directly into bank 1's debt write-down. Consequently, bank 1's network-creditor loss increases linearly with x. For a large loss, $x > \mu(1-\rho)e_1$, bank 1 opts to walk away right after loss x hits. In this situation, bank 1's sponsored creditors seize its collateral, but bank 1's network creditors are also cut loose from the direct loss x. Thus, they are affected only by a flat reputation loss μe_1, which is independent of x, as bank 1 walks away from its sponsored debt obligations.

Figure 5 shows the total losses of bank 1's equity holders, and its network creditors (i.e., banks 2 and 3) for the total (single) loss $\boldsymbol{x}+\boldsymbol{y}$, and the sequential losses $\boldsymbol{x}\to\boldsymbol{y}$. We note that the single

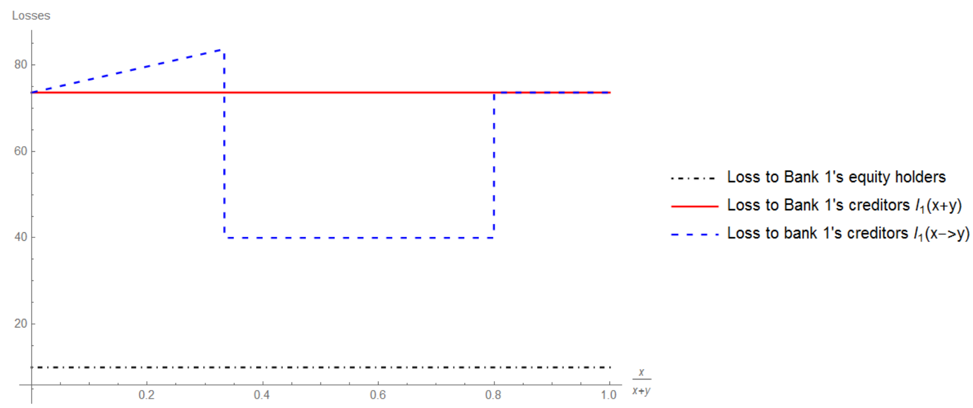

Figure 5: Total loss of bank 1's equity holders (39), and total loss of of bank 1's network creditors (i.e., banks 2 and 3) resulting from the total (single) loss $\boldsymbol{x}+\boldsymbol{y}$, and the sequential losses $\boldsymbol{x}\to\boldsymbol{y}$. The associated banking network is depicted in Figure 1.

cumulative loss M is sufficiently large in this numerical example. Consequently, regardless of how M is partitioned into sequential losses $\boldsymbol{x}\to\boldsymbol{y}$ (such that $x+y=M$), bank 1's equity holders will lose their entire ex-ante value $\overline{w}_1 = 10$ (39) after all losses (and independent of x). In contrast, losses to bank 1's network creditors, under a single loss M and under a loss sequence $\boldsymbol{x}\to\boldsymbol{y}$, differ markedly. In particular, a loss sequence $\boldsymbol{x}\to\boldsymbol{y}$ does not incite a fire sale, and substantially curbs losses for bank 1's network creditors for moderate values of x and y. In our model, it is this loss differential (between the single and sequential loss configurations) that makes government bailout policy relevant to helping banks smooth severe losses.

5 Sequential Losses Hitting Multiple Banks

During crises, losses may hit several banks in the financial system in a relatively short span of time. Therefore, the loss sequencing when losses impact multiple banks is relevant to the financial network. Such sequencing is the subject of the current section.

We consider two losses, $\boldsymbol{x} = (0, \ldots, x, \ldots, 0)^T$ and $\boldsymbol{y} = (0, \ldots, y, \ldots, 0)^T$, that hit banks i and j respectively. Here $x, y > 0$ are the i-th and j-th components of corresponding vectors $\boldsymbol{x}, \boldsymbol{y}$. In case the losses directly hit two different banks (roughly) simultaneously, there is no coordination between these banks in their self-centered responses (stay or walk-away) to losses. In contrast, when losses hit two different banks sequentially, the second bank's response to (and thus, the overall impact of) the second loss depends critically on the earlier response of the first bank to the first loss.[35] In any case, government regulators are presumably aware of these losses (before general markets become aware), and may work with banks i and j separately to minimize possible contagious losses to the financial system. Our analysis in this section indicates that government intervention that synchronizes these losses can be socially beneficial. This analysis thus provides a rationale underlying the Fed's active timing of Lehman Brothers' bankruptcy filings and the announcement of Bank of America's acquisition of Merrill Lynch mentioned in the Introduction.

As in Section 4, for tractability, we maintain assumption (14), $\boldsymbol{c} = \boldsymbol{d} = \boldsymbol{e}$, throughout. It will be clear below that if losses x_i and y_i are large enough, both banks will walk away from their collateralized debt obligations and will retain a positive amount of external assets on their balance sheets. This means that both banks i and j can always avoid an outright default, which in turns implies that no subsequent outright defaults occur. As a result, the relative liability matrix $\boldsymbol{A}$ does not change further after markets clear upon original losses.

5.1 Sequential Losses Hitting Multiple Banks

We first consider the sequence of losses in which x hits bank i first, then y hits bank j. By virtue of the self-centered decision-making protocol (17), when loss x is such that $x \leq (1-\rho)\mu e_i$, bank i stays with its sponsored debt obligations. Otherwise it walks away. Then markets clear contingent on the decision (stay or walk-away) taken by bank i. Next, when bank j is hit with loss y, its

[35] As Section 4 shows, how a bank strategically responds to its own loss affects the balance sheets of other banks in the network. In the setting of sequential losses $\boldsymbol{x}\to\boldsymbol{y}$ hitting two banks, how the first bank has responded to its own loss x presents the second bank (and the entire network) with a fait accompli of the financial landscape, which then is taken into account by the second bank when it devises a self-centered response to its own loss y.

decision toward this loss obviously depends on the decision i took earlier. In particular, if i stayed after x, then j stays if $y \leq (1-\rho)\mu e_j$. Otherwise, j walks away after loss y hits. On the other hand, if i walks away after x, its creditors will seize and sell off its collateral. This causes a decline in collateral c_j at any bank $j \neq i$, so that banks' collateral asset vector $\boldsymbol{c}$ is now only worth $(1-\lambda)\boldsymbol{c}$ following the associated fire sale. By virtue of constraint (18), all banks other than i stay after the fire sale. As mentioned earlier, this parametric assumption allows us to focus on the realistic premise that only banks impacted by direct losses choose to default. After liquidations required to make up for i's fire-sale loss, bank k's illiquid asset is $e_k(x) = \left(1 - \frac{\lambda}{1-\rho}\right) e_k$, $\forall k \neq i$. When bank j is subsequently hit with a direct loss y, it stays if $y \leq \mu(1-\rho)e_j(x)$, where $e_j(x)$ is j's illiquid asset after loss x his bank i given above. Therefore, the condition for j to stay after the second loss y is equivalent to $y \leq \mu(1-\rho-\lambda)e_j$. Otherwise, bank j walks away, in which case other banks $k \neq j$ stay and experience a second liquidation and fire-sale loss of $\frac{\lambda c_k}{1-\rho}$ stemming from j's default (assumption (18)).

To summarize, there are four distinct scenarios following a sequence of losses x hitting i, then y hitting j,

$$\begin{cases} \textbf{Case 1a}\text{: If} \quad x \leq (1-\rho)\mu e_i \quad \text{and} \quad y \leq (1-\rho)\mu e_j, & \text{then both banks stay,} \\ \textbf{Case 2a}\text{: If} \quad x \leq (1-\rho)\mu e_i \quad \text{and} \quad y > (1-\rho)\mu e_j, & \text{then } i \text{ stays, } j \text{ walks away,} \\ \textbf{Case 3a}\text{: If} \quad x > (1-\rho)\mu e_i \quad \text{and} \quad y \leq (1-\rho-\lambda)\mu e_j, & \text{then } i \text{ walks away, } j \text{ stays,} \\ \textbf{Case 4a}\text{: If} \quad x > (1-\rho)\mu e_i \quad \text{and} \quad y > (1-\rho-\lambda)\mu e_j, & \text{then both walk away.} \end{cases} \tag{40}$$

It is crucial to observe that losses hitting banks sequentially lead to asymmetric thresholds in banks' responses due the advantages of having hindsight information and observing market pricing reactions to previous losses (already realized). Indeed, facing a direct (second) loss y, the stay/walk-away decision threshold for bank j jumps from $(1-\rho)\mu e_j$ (if bank i has decided to stay) to $(1-\rho-\lambda)\mu e_j$ (if bank i had walked away from previous loss x). The reason underlying this contingent behavior of the second bank j toward its direct loss y is intuitive. If bank i has already walked away from its sponsored debt obligations after first loss x, all other banks $k \neq i$ had to sell portions of their illiquid assets e_k to meet fire-sale losses stemming from i's default. Hence, with "less skin in the game" ($e_j < e_j(x)$), bank j has less reputation concern ($\mu e_j < \mu e_j(x)$), and thus lowers its threshold to walk away should the second loss y hit. Fire-sale losses are therefore an important conduit that makes loss sequencing relevant when losses hit multiple banks. We next quantitatively characterize the above four scenarios.

Case 1a: If $x \leq (1-\rho)\mu e_i$ and $y \leq (1-\rho)\mu e_j$, then illiquid asset values are $\boldsymbol{e}(\boldsymbol{x}\to\boldsymbol{y}) = \boldsymbol{e} - \frac{\boldsymbol{x}+\boldsymbol{y}}{1-\rho}$. The total loss to the system is, $L(\boldsymbol{x}\to\boldsymbol{y}) = \frac{x+y}{1-\rho} + S(\boldsymbol{x}\to\boldsymbol{y})$, where $S(\boldsymbol{x}\to\boldsymbol{y}) \equiv \sum_{i\in\mathcal{N}} \overline{p}_i - p_i(\boldsymbol{x}\to\boldsymbol{y})$ denotes total shortfalls in payments in the financial system.

Case 2a: If $x \leq (1-\rho)\mu e_i$ and $y > (1-\rho)\mu e_j$, then the illiquid asset values of banks i, j, and any other bank k ex-post (after losses) are respectively,

$$e_i(\boldsymbol{x}\to\boldsymbol{y}) = e_i - \frac{x}{1-\rho} - \frac{\lambda c_i}{1-\rho}, \quad e_j(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)e_j, \quad e_k(\boldsymbol{x}\to\boldsymbol{y}) = \left(1 - \frac{\lambda}{1-\rho}\right) e_k, \;\; \forall k \neq i, j,$$

where $\frac{x}{1-\rho}$ is i's loss from the direct hit x and the associated liquidation, $\frac{\lambda c_i}{1-\rho}$ is i's fire-sale losses stemming from j's default on second loss y.[36] $(1-\mu)e_j$ is j's residual asset value net of the reputation loss after j walks away from y, and $\left(1 - \frac{\lambda}{1-\rho}\right) e_k$ is k's residual asset value net of the fire-sale loss. The total loss to the financial system is, $L(\boldsymbol{x}\to\boldsymbol{y}) = \frac{x}{1-\rho} + y + \mu e_j + S(\boldsymbol{x}\to\boldsymbol{y}) + \sum_{k\neq j} \frac{\lambda c_k}{1-\rho}$, similar to (28).

Case 3a: If $x > (1-\rho)\mu e_i$ and $y \leq (1-\rho-\lambda)\mu e_j$, then the illiquid asset values of banks i, j, and any other bank k ex-post (after losses) are respectively,[37]

$$e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)e_i, \quad e_j(\boldsymbol{x}\to\boldsymbol{y}) = e_j - \frac{y}{1-\rho} - \frac{\lambda c_j}{1-\rho}, \quad e_k(\boldsymbol{x}\to\boldsymbol{y}) = \left(1 - \frac{\lambda}{1-\rho}\right) e_k, \;\; \forall k \neq i, j.$$

The total loss to the system is, $L(\boldsymbol{x}\to\boldsymbol{y}) = \frac{y}{1-\rho} + x + \mu e_i + S(\boldsymbol{x}\to\boldsymbol{y}) + \sum_{k\neq i} \frac{\lambda c_k}{1-\rho}$.

Case 4a: If $x > (1-\rho)\mu e_i$ and $y > (1-\rho-\lambda)\mu e_j$, then the illiquid asset values of banks i, j, and any other bank k (after losses) are respectively,[38]

$$e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)e_i, \quad e_j(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)\left(1 - \frac{\lambda}{1-\rho}\right) e_j, \quad e_k(\boldsymbol{x}\to\boldsymbol{y}) = \left(1 - \frac{2\lambda}{1-\rho}\right) e_k, \;\; \forall k \neq i, j.$$

[36] Observe that bank i does not walk away after bank j's default on loss y. Indeed, constraint (18) implies that $\mu e_i - \mu^2 e_i > \frac{\lambda e_i}{1-\rho}$, while the current **case 2a**: $\mu(1-\rho)e_i > x$ implies $\mu e_i - \frac{\mu x}{1-\rho} > \mu e_i - \mu^2 e_i$ Combining these two inequalities yields $\mu e_i - \frac{\mu x}{1-\rho} > \frac{\lambda e_i}{1-\rho}$, or equivalently $\mu e_i(x) = \mu\left(e_i - \frac{x}{1-\rho}\right) > \frac{\lambda c_i}{1-\rho}$ using (14). Therefore, bank i (which had experienced and stayed after the first loss x) stays with its sponsored debt obligations after bank j walks away upon the second loss y.

[37] Observe that no bank $k \neq i$ walks away after bank i's default on first loss x, because (18) implies that $\mu e_k > \frac{\lambda c_k}{1-\rho}$. Furthermore, $\mu \in (0,1)$, (14) and (18) also assure that $\mu e_k(x) = \mu\left(1 - \frac{\lambda}{1-\rho}\right) e_k > \frac{\lambda c_k}{1-\rho}$, so no bank $k \neq i$ walks away after the second loss y hits j either.

[38] After bank i walks away from its sponsored debt obligations following the first shock x, its under-performing assets (also, its collateral) are seized. Therefore i is not directly affected by a fire sale as j walks away following the second shock y. Furthermore, since $\mu \leq 1$, condition (18) also implies that $\frac{\lambda c_k}{1-\rho} < \mu\left(e_k - \frac{\lambda c_k}{1-\rho}\right)$, $\forall k \neq i, j$, so no bank that is not directly hit by losses x, y walks away from its sponsored debt obligations as a result of the fire sales ignited by i's and j's defaults.

The total loss to the system is, $L(\boldsymbol{x}\to\boldsymbol{y}) = \mu e_i + \mu\left(1-\frac{\lambda}{1-\rho}\right)e_j + x + y + \frac{\lambda c_j}{1-\rho} + S(\boldsymbol{x}\to\boldsymbol{y}) + \sum_{k\neq i,j}\frac{2\lambda c_k}{1-\rho}$. In particular, term $\frac{\lambda c_j}{1-\rho}$ accounts for the liquidation loss to bank j as it had chosen to stay after the first loss x hit bank i. Other terms are similar to those in (28). Before analyzing these results in detail, we present quantitative results for the cases in which losses hit banks simultaneously.

5.2 Simultaneous Losses Hitting Multiple Banks

We now consider the setting in which loss x hits bank i and loss y hits bank j (roughly) simultaneously. As discussed in Section 3 above, in such a setting, neither i nor j has timely information about the other's response to losses (stay or walk away).[39] Both banks then make their decisions independently. In particular, i's decision and j's decision reflect their respective perspectives on their own losses, as follows.

$$\begin{cases} \textbf{Case 1b: If} \quad x \leq (1-\rho)\mu e_i \quad \text{and} \quad y \leq (1-\rho)\mu e_j, & \text{then both banks stay,} \\ \textbf{Case 2b: If} \quad x \leq (1-\rho)\mu e_i \quad \text{and} \quad y > (1-\rho)\mu e_j, & \text{then } i \text{ stays, } j \text{ walks away,} \\ \textbf{Case 3b: If} \quad x > (1-\rho)\mu e_i \quad \text{and} \quad y \leq (1-\rho)\mu e_j, & \text{then } i \text{ walks away, } j \text{ stays,} \\ \textbf{Case 4b: If} \quad x > (1-\rho)\mu e_i \quad \text{and} \quad y > (1-\rho)\mu e_j, & \text{then both walk away.} \end{cases} \tag{41}$$

It is crucial to observe that losses hitting banks simultaneously produce symmetric thresholds in those banks' responses due to the lack of coordination and real-time information. This is in sharp contrast with the asymmetric thresholds (40) in banks' responses to sequential losses, and underlies the potential social benefits of government intervention to time banks' severe losses. We quantitatively characterize the above four scenarios below.

Case 1b: If $x \leq (1-\rho)\mu e_i$ and $y \leq (1-\rho)\mu e_j$, both i and j stay with their sponsored debt obligations. Banks' illiquid assets (after losses) are $\boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{e} - \frac{\boldsymbol{x}+\boldsymbol{y}}{1-\rho}$. Only direct losses, the associated liquidation costs, and interbank shortfalls contribute to the total loss of the financial system, $L(\boldsymbol{x}+\boldsymbol{y}) = \frac{x+y}{1-\rho} + S(\boldsymbol{x}+\boldsymbol{y})$.

Case 2b: If $x \leq (1-\rho)\mu e_i$ and $y > (1-\rho)\mu e_j$, i stays with, while j walks away, from their

[39] It might happen that bank j has an interbank claim on bank i. In this case, i's losses will transmit to j, and j will take into account this possibility when making its stay/walk-away decision. Our framework allows for such calculation by j.

sponsored debt obligations. Banks' illiquid assets (after losses) are,

$$e_i(\boldsymbol{x}+\boldsymbol{y}) = e_i - \frac{x}{1-\rho} - \frac{\lambda c_i}{1-\rho}, \qquad e_j(\boldsymbol{x}+\boldsymbol{y}) = (1-\mu)e_j, \qquad e_k(\boldsymbol{x}+\boldsymbol{y}) = \left(1 - \frac{\lambda}{1-\rho}\right)e_k, \ \ \forall k \neq i, j,$$

and the total loss to the financial system is, $L(\boldsymbol{x}+\boldsymbol{y}) = \frac{x}{1-\rho} + y + \mu e_j + S(\boldsymbol{x}+\boldsymbol{y}) + \sum_{k \neq j} \frac{\lambda c_k}{1-\rho}$.

Case 3b: If $x > (1-\rho)\mu e_i$ and $y \leq (1-\rho)\mu e_j$, i walks away from, while j stays with, their sponsored debt obligations. Banks' illiquid assets (after losses) are,

$$e_i(\boldsymbol{x}+\boldsymbol{y}) = (1-\mu)e_i, \qquad e_j(\boldsymbol{x}+\boldsymbol{y}) = e_j - \frac{y}{1-\rho} - \frac{\lambda c_j}{1-\rho}, \qquad e_k(\boldsymbol{x}+\boldsymbol{y}) = \left(1 - \frac{\lambda}{1-\rho}\right)e_k, \ \ \forall k \neq i, j,$$

and the total loss to the financial system is, $L(\boldsymbol{x}+\boldsymbol{y}) = \frac{y}{1-\rho} + x + \mu e_i + S(\boldsymbol{x}+\boldsymbol{y}) + \sum_{k \neq i} \frac{\lambda c_k}{1-\rho}$.

Case 4b: If $x > (1-\rho)\mu e_i$ and $y > (1-\rho)\mu e_j$, then banks' illiquid assets (after losses) are,

$$e_i(\boldsymbol{x}+\boldsymbol{y}) = (1-\mu)e_i, \qquad e_j(\boldsymbol{x}+\boldsymbol{y}) = (1-\mu)e_j, \qquad e_k(\boldsymbol{x}+\boldsymbol{y}) = \left(1 - \frac{2\lambda}{1-\rho}\right)e_k, \ \ \forall k \neq i, j,$$

and the total loss to the financial system is, $L(\boldsymbol{x}+\boldsymbol{y}) = x + y + \mu e_i + \mu e_j + S(\boldsymbol{x}+\boldsymbol{y}) + \sum_{k \neq i,j} \frac{2\lambda c_k}{1-\rho}$.

Discussion

Observe that the functional form of the total loss $L(\boldsymbol{x}+\boldsymbol{y})$ for the simultaneous-loss configuration is quite similar to the total loss $L(\boldsymbol{x} \to \boldsymbol{y})$ for the sequential-loss configuration (see **Cases 1a-4a** (40) compared to the corresponding cases **Cases 1b-4b** (41)).

To contrast these total losses, we start by examining the ex-post (after losses) illiquid asset values $e(\boldsymbol{x} \to \boldsymbol{y})$ as compared to $e(\boldsymbol{x}+\boldsymbol{y})$. Following from the quantitative characterization in Sections 5.1 and 5.2, the ex-post illiquid asset values differ in sequential-loss as opposed to simultaneous-loss configurations if and only if the losses (x, y) are in the following two regions,[40]

$$\begin{aligned} \Gamma_{4a,3b} &\equiv \{(x,y) : [x > \mu(1-\rho)e_i] \ \cap \ [\mu(1-\rho)e_j > y > \mu(1-\rho-\lambda)e_j]\} \\ \Gamma_{4a,4b} &\equiv \{(x,y) : [x > \mu(1-\rho)e_i] \ \cap \ [y > \mu(1-\rho)e_j]\} \end{aligned} \tag{42}$$

[40]For all other loss values, market clearings, prices and losses are independent of the loss sequencing; $\mathbf{p}(\boldsymbol{x} \to \boldsymbol{y}) = \mathbf{p}(\boldsymbol{x}+\boldsymbol{y})$, $e_i(\boldsymbol{x} \to \boldsymbol{y}) = e_i(\boldsymbol{x}+\boldsymbol{y})$, $l_i(\boldsymbol{x} \to \boldsymbol{y}) = l_i(\boldsymbol{x}+\boldsymbol{y})$, $\forall (x,y) \notin \{\Gamma_{4a,3b} \cup \Gamma_{4a,4b}\}$.

Region $\Gamma_{4a,4b}$: For losses $(x, y) \in \Gamma_{4a,4b}$, both banks walk away from their sponsored businesses and the illiquid asset values ex-post (after losses) are as follows:

$$\begin{cases} e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)e_i = e_i(\boldsymbol{x}+\boldsymbol{y}), \\ e_j(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)\left(1-\frac{\lambda}{1-\rho}\right)e_j < (1-\mu)e_j = e_j(\boldsymbol{x}+\boldsymbol{y}), \\ e_k(\boldsymbol{x}\to\boldsymbol{y}) = \left(1-\frac{2\lambda}{1-\rho}\right)e_k = e_k(\boldsymbol{x}+\boldsymbol{y}), \ \ \forall k \neq i, j. \end{cases} \tag{43}$$

In this region, bank j's illiquid assets have higher value in the simultaneous-loss configuration than they do in the sequential-loss configuration. This is because in the sequential-loss configuration, bank j has already had less illiquid assets (due to its liquidation related to i's default) before loss y hits. As a result, when j walks away from the second loss y, j also ends up with proportionally less illiquid assets after loss sequence $\boldsymbol{x}\to\boldsymbol{y}$ (compared to the simultaneous-loss configuration). Similarly,

$$\mu\left(1-\frac{\lambda}{1-\rho}\right)e_j + \frac{\lambda e_j}{1-\rho} > \mu e_j,$$

or the direct loss incurred to bank j is always larger in the case of sequential losses (left-hand side) than in case of simultaneous losses (right-hand side). In fact, the same is true for the total loss to the system. More colloquially, in the region $\Gamma_{4a,4b}$, a cascade of dominoes imposes a higher eventual total loss than an earthquake.

Proposition 8 *When shocks hitting banks i, j are in region $\Gamma_{4a,4b}$ (42), the total loss to the financial system is always larger when these shocks arrive sequentially than when they arrive simultaneously,*

$$L(\boldsymbol{x}\to\boldsymbol{y}) > L(\boldsymbol{x}+\boldsymbol{y}), \qquad \forall (x, y) \in \Gamma_{4a,4b}.$$

Together with (43), this result lends support to the government's attempt to synchronize losses hitting different banks to curb losses to individual banks as well as to the entire financial system. In the real world, such an effort was illustrated by the government's timing of Lehman Brothers' bankruptcy filings and the announcement of Bank of America's acquisition of Merrill Lynch mentioned in the Introduction. In our framework, the loss timing works if losses are sufficiently large to prompt the defaults at two or more banks ($(x, y) \in \Gamma_{4a,4b}$). Then simultaneous defaults limit losses and thus improve welfare compared to sequential losses. This is because the defaulting banks are able to walk away from under-performing assets before fire sales resulting from other banks' defaults inflict losses on their balance sheets by devaluing some of their assets.

Proof. If $(x, y) \in \Gamma_{4a,4b}$, then $\boldsymbol{e}(\boldsymbol{x}\to\boldsymbol{y}) < \boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y})$ (43), and so for all $\boldsymbol{p}$, $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{e}, \boldsymbol{b}, \boldsymbol{x}\to\boldsymbol{y}) = \min\{A^T\boldsymbol{p} + \boldsymbol{e}(\boldsymbol{x}\to\boldsymbol{y}), \overline{\boldsymbol{p}}\} < \min\{A^T\boldsymbol{p} + \boldsymbol{e}(\boldsymbol{x}+\boldsymbol{y}), \overline{\boldsymbol{p}}\} = \Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{e}, \boldsymbol{b}, \boldsymbol{x}+\boldsymbol{y})$. As a result, $\boldsymbol{p}(\boldsymbol{x}\to\boldsymbol{y}) < \boldsymbol{p}(\boldsymbol{x}+\boldsymbol{y})$, which then implies that $S(\boldsymbol{x}\to\boldsymbol{y}) = \sum_{i=1}^{n}(\overline{p}_i - p_i(\boldsymbol{x}\to\boldsymbol{y})) > \sum_{i=1}^{n}(\overline{p}_i - p_i(\boldsymbol{x}+\boldsymbol{y})) = S(\boldsymbol{x}+\boldsymbol{y})$. Therefore we have

$$
\begin{aligned}
L(\boldsymbol{x}\to\boldsymbol{y}) &= \mu e_i + \mu\left(1 - \frac{\lambda}{1-\rho}\right)e_j + x + y + S(\boldsymbol{x}\to\boldsymbol{y}) + \frac{\lambda c_j}{1-\rho} + \sum_{k\neq i,j}\frac{2\lambda c_k}{1-\rho} \\
&> \mu e_i + \mu e_j + x + y + S(\boldsymbol{x}+\boldsymbol{y}) + \sum_{k\neq i,j}\frac{2\lambda c_k}{1-\rho} = L(\boldsymbol{x}+\boldsymbol{y}),
\end{aligned}
$$

for any losses $(x, y) \in \Gamma_{4a,4b}$ ■

Region $\Gamma_{4a,3b}$: For losses $(x, y) \in \Gamma_{4a,3b}$ specified in (42), both banks i and j walk away from their sponsored debt obligations when losses hit sequentially ($\boldsymbol{x}\to\boldsymbol{y}$, see **Case 4a**), while i walks away and j stays when losses hit simultaneously ($\boldsymbol{x}+\boldsymbol{y}$, see **Case 3b**). In this region $\Gamma_{4a,3b}$, loss x is severe enough to prompt i to walk away immediately. Whereas, loss y is marginal; (i) it is larger than j's reduced reputation/liquidation loss[41] ($y > \mu(1-\rho-\lambda)e_j$, so j walks away if x and y hit sequentially as in **Case 4a**), but (ii) it is smaller than j's full reputation/liquidation loss[42] ($y < \mu(1-\rho)e_j$, so j stays if x and y hit simultaneously as in **Case 3b**).

Comparing **Case 4a** and **Case 3b** yields,

$$
\begin{cases}
e_i(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)e_i = e_i(\boldsymbol{x}+\boldsymbol{y}), \\
e_j(\boldsymbol{x}\to\boldsymbol{y}) = (1-\mu)\left(1 - \frac{\lambda}{1-\rho}\right)e_j > e_j - \frac{y+\lambda e_j}{1-\rho} = e_j(\boldsymbol{x}+\boldsymbol{y}), \\
e_k(\boldsymbol{x}\to\boldsymbol{y}) = \left(1 - \frac{2\lambda}{1-\rho}\right)e_k < \left(1 - \frac{\lambda}{1-\rho}\right)e_k = e_k(\boldsymbol{x}+\boldsymbol{y}), \quad \forall k \neq i, j.
\end{cases}
\tag{44}
$$

Intuitively, when the loss x directly hitting bank i is sufficiently large, i walks away from its sponsored debt obligation whether losses hit simultaneously or sequentially. Therefore, i's illiquid asset value does not depend on the loss sequencing, $e_i(\boldsymbol{x}\to\boldsymbol{y}) = e_i(\boldsymbol{x}+\boldsymbol{y})$. In contrast, bank j's illiquid assets have higher value in the sequential-loss configuration than in the simultaneous-loss configuration, $e_j(\boldsymbol{x}\to\boldsymbol{y}) > e_j(\boldsymbol{x}+\boldsymbol{y})$.[43] That is because by walking away from y (in the sequential-loss configuration, **Case 4a**), bank j relieves itself from the direct loss y (compared to j's internalizing

[41] The reduction is a result of the fire sale ignited when bank i walks away earlier (from the first loss x in the sequence $\boldsymbol{x}\to\boldsymbol{y}$) in **Case 4a**.

[42] The reputation/liquidation loss to j is complete because j's asset value is not affected by i's default on (walk-away from) a simultaneous loss x in **Case 3b**.

[43] Technically, this is a consequence of the bound $y > \mu(1-\rho-\lambda)e_j$ specified in $\Gamma_{4a,3b}$ (42).

this loss in the simultaneous-loss configuration, **Case 3b**), which is sizable in region $\Gamma_{4a,3b}$. Finally, for banks k not directly hit by either loss x or loss y, the illiquid asset values are lower in the sequential-loss configuration ($e_k(\boldsymbol{x}\rightarrow\boldsymbol{y}) < e_k(\boldsymbol{x}+\boldsymbol{y})$, $\forall k \neq i, j$), because they are hit by two fire sales (associated with i's and j's defaults) in that configuration.

In sum, when loss values are in region $\Gamma_{4a,3b}$, loss synchronization (i) does not affect bank i's illiquid asset values, but (ii) decreases bank j's illiquid asset values, and (iii) increases all other bank k's illiquid asset values. These results indicate that when loss values are in region $\Gamma_{4a,3b}$, the government's synchronizing of losses hitting different banks does not always benefit every bank in the financial system. However, for certain numerical loss values in $\Gamma_{4a,3b}$, when we take into account the market-clearing effects, losses in banks $k \neq i, j$ can impose further losses back onto banks i, j through interbank linkages such that the eventual (ex-post) equity value at every bank in the network is higher in the simultaneous-loss than in the sequential-loss configuration. In this situation, the government's timing of losses proves to be socially beneficial from bank equity holders' perspective. Our numerical example below demonstrate such a circumstance.

5.3 Numerical Illustration

For a numerical illustration of the analysis presented in Sections 5.1-5.2, consider a stylized financial network of three banks, whose interbank and outside holdings are shown in Figure 6. Consider two

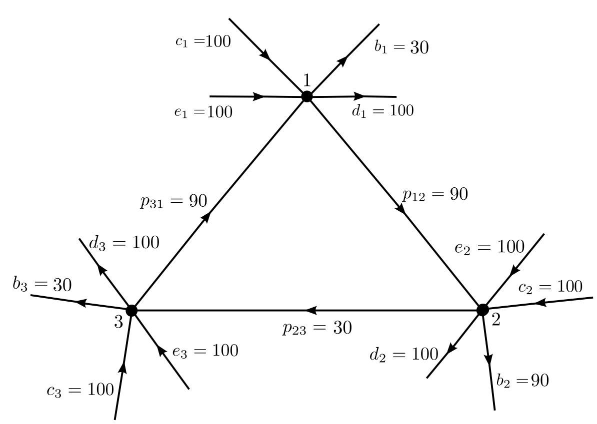

Figure 6: A network with three banks. Incoming arrows to and outgoing arrows from a bank respectively depict assets and liabilities of that bank. The liquidation, reputation and fire-sale parameters respectively are $\rho = 0.05$, $\mu = 0.5$ and $\lambda = 0.23$.

losses $\boldsymbol{x}, \boldsymbol{y}$ that respectively hit bank 1 (or i) and 2 (or j); $\boldsymbol{x} = (x, 0, 0)$, $\boldsymbol{y} = (0, y, 0)$. Ex-post liabilities (i.e., clearing payment vector) and all other quantities at market clearings are computed on the basis of the numerical solution to the unique fixed-point of the map (21). Figure 7 plots

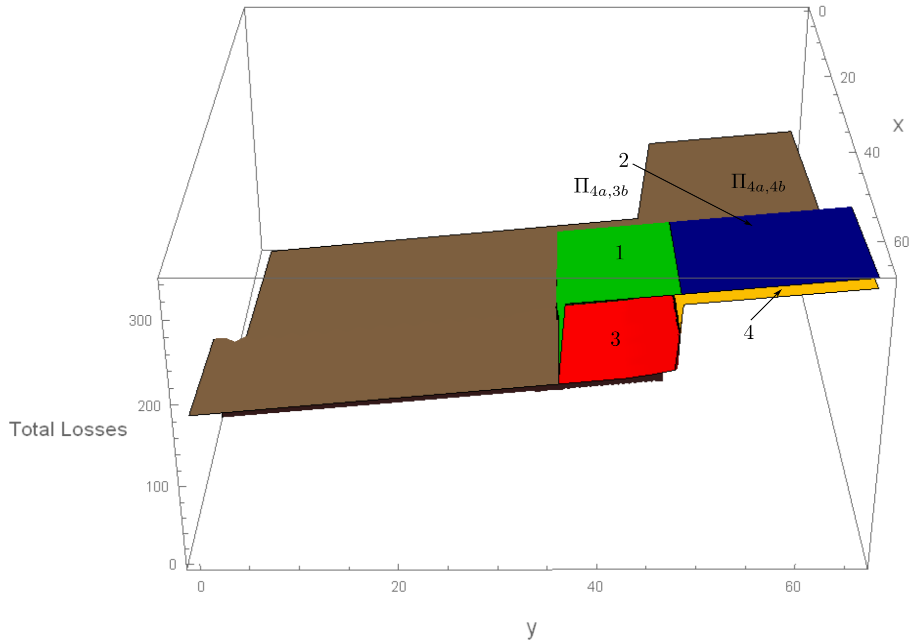

Figure 7: Total losses $L(\boldsymbol{x}\to\boldsymbol{y})$ (blue (numbered 2), green (numbered 1) and brown (not numbered) patches) for sequential losses $\boldsymbol{x}\to\boldsymbol{y}$, and $L(\boldsymbol{x}+\boldsymbol{y})$ (yellow (numbered 4), red (numbered 3) and brown (not numbered) patches) for simultaneous losses $\boldsymbol{x}+\boldsymbol{y}$, plotted against loss values (x, y). The underlying financial network is specified as in Figure 6.

the total losses $L(\boldsymbol{x}\to\boldsymbol{y})$ (for the sequential-loss configuration) and $L(\boldsymbol{x}+\boldsymbol{y})$ (for the simultaneous-loss configuration) to the the network versus loss values (x, y). To contrast these total losses, (i) blue (numbered 2) and yellow (numbered 4) patches depict respectively $L(\boldsymbol{x}\to\boldsymbol{y})$ and $L(\boldsymbol{x}+\boldsymbol{y})$ for $(x, y) \in \Gamma_{4a,4b}$, (ii) green (numbered 1) and red (numbered 3) patches depict respectively $L(\boldsymbol{x}\to\boldsymbol{y})$ and $L(\boldsymbol{x}+\boldsymbol{y})$ for $(x, y) \in \Gamma_{4a,3b}$, and (iii) brown (not numbered) patches for all other loss values. Observe that blue (numbered 2) is above the yellow (numbered 4) patch, green (numbered 1) is above the red (numbered 3) patch, and brown (not numbered) patches coincide. Therefore, in this numerical example, compared to total loss $L(\boldsymbol{x}+\boldsymbol{y})$, total loss $L(\boldsymbol{x}\to\boldsymbol{y})$ is higher in regions $\Gamma_{4a,4b}$ and $\Gamma_{4a,3b}$, and is the same for all other loss values. These numerical patterns confirm Proposition 8, (43) and (44). To understand the incentive compatibility constraint of bank 2 (which suffers direct loss y) toward the possible government's action to time the losses, Figure 8 plots bank 2's ex-post equity value (i.e., net worth) $w_2(\boldsymbol{x}\to\boldsymbol{y})$ (for the sequential-loss configuration) and $w_2(\boldsymbol{x}+\boldsymbol{y})$ (for the simultaneous-loss configuration) versus loss values (x, y).[44] To contrast these net worths,

[44] Recall that w's and $\overline{w}$'s respectively denote the ex-post (after losses) and ex-ante equity values. The former has been introduced and employed in (39). Here, concerning Figure 8, the ex-post equity value (or net worth)

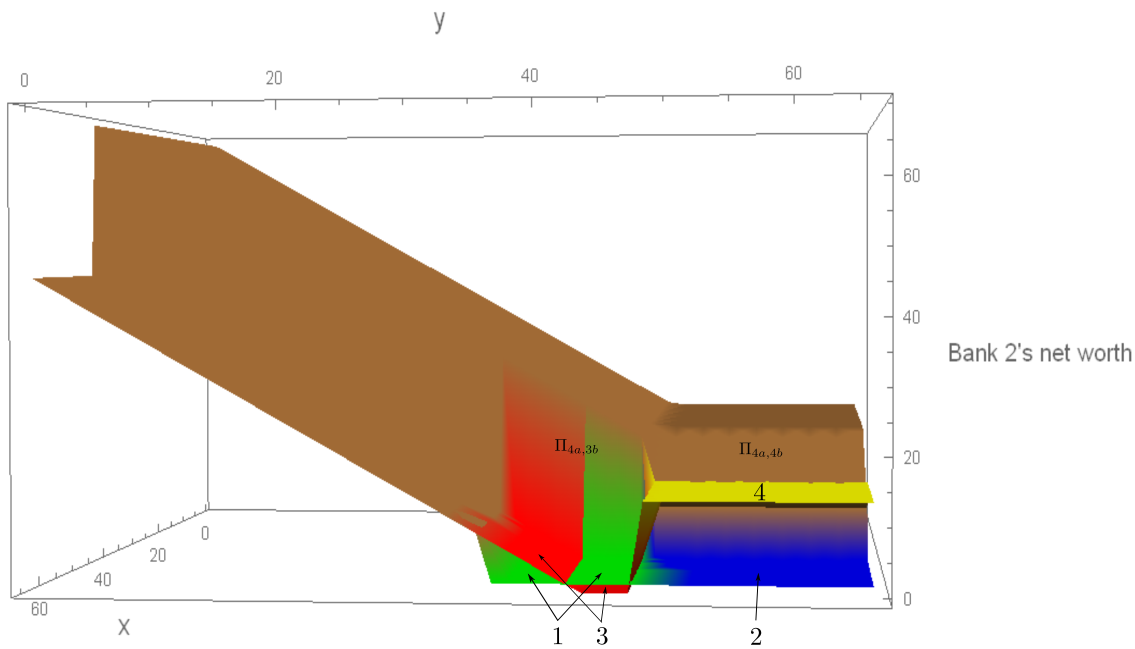

Figure 8: Bank 2's net worth (i.e., equity value) $w_2(\boldsymbol{x}{\to}\boldsymbol{y})$ (blue (numbered 2), green (numbered 1) and brown (not numbered) patches) for sequential losses $\boldsymbol{x}{\to}\boldsymbol{y}$, and $w_2(\boldsymbol{x}{+}\boldsymbol{y})$ (yellow (numbered 4), red (numbered 3) and brown (not numbered) patches) for simultaneous losses $\boldsymbol{x}{+}\boldsymbol{y}$, plotted against loss values (x, y). The underlying financial network is specified as in Figure 6.

(i) blue (numbered 2) and yellow (numbered 4) patches depict respectively $w_2(\boldsymbol{x}{\to}\boldsymbol{y})$ and $w_2(\boldsymbol{x}{+}\boldsymbol{y})$ for losses $(x, y) \in \Gamma_{4a,4b}$, (ii) green (numbered 1) and red (numbered 3) patches depict respectively $w_2(\boldsymbol{x}{\to}\boldsymbol{y})$ and $w_2(\boldsymbol{x}{+}\boldsymbol{y})$ for losses $(x, y) \in \Gamma_{4a,3b}$, and (iii) brown (not numbered) patches for all other loss values. Observe that while the blue patch is below the yellow patch (and the brown patches coincide), the green (numbered 1) patch is not necessarily always below the red (numbered 3) patch. Therefore, in this numerical example, bank 2's ex-post equity value $w_2(\boldsymbol{x}{\to}\boldsymbol{y})$ is lower in regions $\Gamma_{4a,4b}$, but not necessarily always lower in region $\Gamma_{4a,3b}$, compared to equity value $w_2(\boldsymbol{x}{+}\boldsymbol{y})$ (while $w_2(\boldsymbol{x}{\to}\boldsymbol{y}) = w_2(\boldsymbol{x}{+}\boldsymbol{y})$ for all other loss values). Consequently, the government's attempt to synchronize sequential losses $\boldsymbol{x}{\to}\boldsymbol{y}$ might or might not necessarily be in the best interest of the equity holders of bank 2, who suffer the second loss of the sequence, for some loss values.[45] Elsewhere in this numerical example, such government intervention proves socially beneficial and

vector of banks are given as follows, $\boldsymbol{w}(\boldsymbol{x}{+}\boldsymbol{y}) = \max\{A^T\boldsymbol{p}(\boldsymbol{x}{+}\boldsymbol{y}) + \boldsymbol{e}(\boldsymbol{x}{+}\boldsymbol{y}) - \boldsymbol{p}(\boldsymbol{x}{+}\boldsymbol{y}), 0\}$ for simultaneous losses, and $\boldsymbol{w}(\boldsymbol{x}{\to}\boldsymbol{y}) = \max\{A^T\boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y}) + \boldsymbol{e}(\boldsymbol{x}{\to}\boldsymbol{y}) - \boldsymbol{p}(\boldsymbol{x}{\to}\boldsymbol{y}), 0\}$ for sequential losses. Note that in these formulas, any bank k's holding of sponsored business c_k and liability d_k are not present. This is because if bank k chooses to stay after the second shock, it must post a sufficient amount of collateral to match its liability, i.e., $c_k = d_k$, and these cancel each other in the calculation of net worth. If bank k chooses to walk away, the collateral is seized by the creditors, and this clears k's liability b_k.

[45]These loss values are such that the green (numbered 1) patch lies above the red (numbered 3) patch in Figure 8.

desirable in our stylized model of the financial network.

6 Conclusion

In a stylized banking system, as we demonstrate, a single large loss has the potential to leave markedly different impacts on the financial system than does a series of moderate losses of the same cumulative magnitude. This relevance result on loss sequencing arises because although banks strategically bail out lenders of their under-performing businesses based on myopic losses, the eventual losses are determined as well by subsequent defaults, fire sales and other network effects – factors that are practically infeasible for banks to compute ex-ante.

Our findings on the differences between simultaneous and sequential losses – or colloquially, between earthquakes and tumbling dominoes – have significant implications for government bailout policies. Given some underlying structures of claims of banks and outside institutions on one another, the government may be able to reduce losses by taking regulatory actions to engineer a string of losses into a single large event. Under other structures, government measures have the potential to reduce losses by bailing out banks that are threatened early on, and thereby turning a major loss into a sequence of smaller losses.

This analysis identified the conditions when each of these two polar opposite government policies is desirable. In the 2008 financial meltdown, the government implicitly recognized the need to save some institutions, while forcing others to immediately absorb losses and accept fire sales. Such apparently inconsistent policies might represent sensible government policy. Our analysis was a conceptual formulation. Whether the government correctly diagnosed the conditions for the two actions in 2008 will be long debated.

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Appendices

A Two Historical Events and Excerpts

Lehman Brothers' Bankruptcy Filing

The following excerpt is from The Financial Crisis Inquiry Commission (2011) (FCIC), recounting an encounter between Lehman Brothers' management team and New York Fed and SEC authorities right before Lehman Brothers filed for bankruptcy. It refers to a meeting on Sunday, September 14, 2008, between Lehman's team (President Bart McDade, CFO Ian Lowitt, Head of Principal Investing Alex Kirk, Lehman's bankruptcy counsel Harvey Miller) and regulators (NY Fed General Counsel Tom Baxter, then-head of NY Fed's Markets Group William Dudley, and director of the SECs Division of Trading and Markets Erik Sirri). This meeting took place right after Lehman's team learned that NY Fed would provide more flexible terms for the Primary Dealer Credit Facility (PDCF) lending facility, which would include expanding the types of collateral borrowers could use.

" . . . during that Sunday meeting at the New York Fed, government officials stepped out for an hour and came back to ask: "Are you planning on filing bankruptcy tonight?" A surprised Miller replied that "no one in the room was authorized to file the company, only the Board could . . . " Unmoved, government officials explained that directors of Lehman's U.K. subsidiary – LBIE – would be personally liable if they did not file for bankruptcy by the opening of business Monday. As Kirk recalled, "They then told us 'we would like you to file tonight . . . It's the right thing to do, because there's something else which we can't tell you that will happen this evening. We would like both events to happen tonight before the opening of trading Monday morning.' " The second event would turn out to be the announcement of Bank of America's acquisition of Merrill Lynch."

In the end, Lehman Brothers filed for Chapter 11 bankruptcy protection on September 15, 2008, in full compliance with regulators' timing suggestion.

Bear Stearns' Bailout Attempt

The following excerpt is from The New York Times (June 23, 2007) article, recounting the enormous and highly unusual bailout decision by Bear Stearns towards one of the hedge funds under its management. This article refers to the up-to $3.2 billion in loans pledged by Bear Stearns on June 22, 2007 to bail out its hedge fund named "Bear Stearns High-Grade Structured Credit Fund".

The fund was collapsing because of its bad bets on sub-prime mortgages. The highly leveraged fund was set up in 2004, financed by billions of dollars borrowed from other banks and brokerage firms, and mere tens of millions of dollars from Bear Stearns.

" . . . The bailout was a major departure for the firm, which has long resisted putting too much of its own capital at risk. But in this case, the stakes were too high. If lenders had seized the assets of the funds and tried to sell billions of dollars in mortgage-related securities at fire-sale prices, it could have exposed Bear Stearns and the market to substantial losses. . . . 'Bear Stearns is bailing one of the funds out because it is worried about the damage to its reputation if it stuck investors and lenders with big losses,' said Dick Bove, an analyst with Punk Ziegel & Company. 'If they walked away from it, investors would have lost all their money and lenders would have lost all of the money,' Mr. Bove said. But 'if they did that to everyone in the financial community, the financial community would have shut them down.' "

In the end, the bailout attempt fell through because another hedge fund under Bear Stearns' sponsorship also incurred huge losses such that bailout option became impossible. Bear Stearns itself collapsed quickly afterward and was sold to JPMorgan Chase in March 2008 in a bailout orchestrated by the Fed.

B Technical Derivations and Proofs

B.1 Eisenberg-Noe's Setting

We explain the difference between Eisenberg and Noe (2001)'s setting and ours, which follows Glasserman and Young (2015).The definition of Eisenberg-Noe's Φ is

$$\Phi(\boldsymbol{p}; \boldsymbol{\Pi}, \overline{\boldsymbol{p}}, \boldsymbol{e}) = \min\{\boldsymbol{\Pi}^T \boldsymbol{p} + \boldsymbol{e}, \overline{\boldsymbol{p}}\},$$

where $\boldsymbol{\Pi}$ is the counterpart of the relative liability matrix $\boldsymbol{A}$ in this paper, where the external liabilities $\boldsymbol{b}$ is set to equal $\mathbf{0}$. As a consequence, $\sum_{j=1}^{n} \Pi_{ij} = 1$, and hence $\rho(\boldsymbol{\Pi}) = 1$ so one cannot use the Banach fixed-point theorem to show the uniqueness of the clearing payment vector more involved. Eisenberg-Noe's proof of uniqueness requires an assumption on $\boldsymbol{e}$, which is the vector of exogenously operating cash flows in their model. To see the nature of the difference between our key assumption (C) for uniqueness (which follows Glasserman and Young (2015)) and theirs, we need to introduce some definitions.

Definition 3 *A set $\mathcal{S} \subset \mathcal{N}$ is called a surplus set if no node in the set has any obligations to any node outside the set and the set has a positive total cash flow.*

Definition 4 *For each node $i \in \mathcal{N}$, define the risk orbit $o(i)$ as the following set*

$$o(i) = \{j \in \mathcal{N} : \textit{there exists a chain of positive liability from } i \textit{ to } j\}$$

Eisenberg-Noe's key assumption for the proof of uniqueness of the clearing payment vector is: Every risk orbit $o(i)$ is a surplus set.

Proof of Proposition 6: *(i)* If bank i walks away, its direct losses must have been sufficiently large (17), $x_i \geq \mu(1-\rho)e_i$. When i walks away, its sponsored creditors $\mathcal{I}$ seize collateral valued at $c_i - x_i$ for original loans of amount d_i. Because $d_i = c_i$ (14), $\mathcal{I}$ suffer losses of x_i, which is at least $\mu(1-\rho)e_i$ by the revealed choice (walk-away) made by bank i. As i walks away from its implicit guarantees, it is neither in possession of sponsored assets $c_i - x_i$ nor obligations d_i, and its (illiquid) assets suffer from a reputation-damage discount and is valued only at $(1-\mu)e_i$. Since $\mu \in (0,1)$ (16), this remaining asset value is still strictly positive. Second result of Proposition 2 then implies that i will not be outright default after markets clear.

(ii) In the opposite scenario, if bank i stays, its direct losses must have been sufficiently small, $x_i < \mu(1-\rho)e_i$. Then i posts additional collateral to boost them up to the original face value of the sponsored debts d_i (14). As a result, i's sponsored creditors $\mathcal{I}$ suffer no losses after markets clear due to their settlement priority with bank i, which take place before those of i's other creditors.

For its part, after posting additional collateral, i's remaining assets are valued at is sufficiently large (equivalent, direct losses to i were sufficiently small for i to stay in the first place); $e_i - \frac{x_i}{1-\rho} > e_i - \mu c_i > 0$.[46]. An application of Proposition 2 implies immediately that i will not be outright default after markets clear ■

B.2 Existence and Uniqueness of the Clearing Payment Vector

Existence of the Clearing Payment Vector

To show existence of the clearing payment vector, first we need the following properties of the map $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) : [\boldsymbol{0}, \overline{\boldsymbol{p}}] \to [\boldsymbol{0}, \overline{\boldsymbol{p}}]$ as defined in (5).

Lemma 1 *(i) $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ is non-decreasing in $\boldsymbol{p}$, and (ii) $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ is non-increasing in $\overline{\boldsymbol{p}}$.*

Proof. (i) If $\boldsymbol{p}' \geq \boldsymbol{p}$ then we have $\Phi(\boldsymbol{p}'; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) = \min\{\boldsymbol{A}^T\boldsymbol{p}' + \boldsymbol{c} - \boldsymbol{x} \ , \ \overline{\boldsymbol{p}}\} \geq \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x} \ , \ \overline{\boldsymbol{p}}\} = \Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$.
(ii) If $\overline{\boldsymbol{p}}' \geq \overline{\boldsymbol{p}}$ then we have $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}', \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) = \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x} \ , \ \overline{\boldsymbol{p}}'\} \geq \min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x} \ , \ \overline{\boldsymbol{p}}\} = \Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$. ■

[46]Recall that when i stays, then after meeting margin call, only assets of amount $e_i - \frac{x_i}{1-\rho}$ are left to i's interbank and other debt holders.

Lemma 2 *For any financial system* $(\boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b})$ *and a loss vector* $\boldsymbol{x}$*, there exists a fixed point* $\boldsymbol{p}(\boldsymbol{x})$ *of the map* $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$*.*

Proof. Consider a sequence $\{\boldsymbol{p}^k(\boldsymbol{x})\}_{k\geq 0}$ defined inductively as follows,

$$\boldsymbol{p}^0(\boldsymbol{x}) = \overline{\boldsymbol{p}}, \qquad \boldsymbol{p}^{i+1}(\boldsymbol{x}) = \Phi(\boldsymbol{p}^i(\boldsymbol{x}); \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) \quad \forall i \geq 0.$$

By Lemma 1 a., we have $\boldsymbol{p}^0(\boldsymbol{x}) \geq \boldsymbol{p}^1(\boldsymbol{x}) \geq \cdots \geq \boldsymbol{p}^k(\boldsymbol{x}) \geq \ldots$. Thus, the series $\{\boldsymbol{p}^k(\boldsymbol{x})\}_{k\geq 0}$ is weakly decreasing and bounded from below by $\boldsymbol{0}$. It follows that there exists $\boldsymbol{p}(\boldsymbol{x}) = \lim_{k\to\infty} \boldsymbol{p}^k(\boldsymbol{x})$ ■

Uniqueness of the Clearing Payment Vector.

$\mathfrak{C}$ is the key sufficient condition for the uniqueness of the clearing payment vector $\boldsymbol{p}(\boldsymbol{x})$. We first show that if the financial system $(\boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b})$ satisfies $\mathfrak{C}$ then it is true that $\boldsymbol{A}$ has spectral radius less than 1. To prove this claim, we need the following auxiliary lemmas.

Lemma 3 *If the sum of entries on each row of a matrix* $\boldsymbol{A}$ *is strictly less than 1 then its spectral radius* $\rho(\boldsymbol{A})$ *is less than 1.*

Proof. We use the maximum norm for vectors: if $\boldsymbol{v} = (v_1, \ldots, v_n)$ then $\|\boldsymbol{v}\| = \max\{|v_1|, |v_2|, \ldots |v_n|\}$. It's easy to see that the corresponding subordinate matrix norm is $\|\boldsymbol{A}\| := \sup_{\|\boldsymbol{v}\|=1} \|\boldsymbol{A}\boldsymbol{v}\| = \max\{|r_1|, |r_2|, \ldots |r_n|\} < 1$ where r_i is the sum of entries on row i. Now if λ is any eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{v}$ is an eigenvector for λ then we have

$$|\lambda|\|\boldsymbol{v}\| = \|\lambda\boldsymbol{v}\| = \|\boldsymbol{A}\boldsymbol{v}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{v}\|.$$

Hence, we have $|\lambda| \leq \|\boldsymbol{A}\| < 1$. Since this is true for any eigenvalue of $\boldsymbol{A}$, we have $\rho(\boldsymbol{A}) < 1$ ■

Lemma 4 *Let* A *be a square matrix and* $\rho(\boldsymbol{A})$ *be its spectral radius. Then* $\rho(\boldsymbol{A}) < 1$ *if and only if* $\lim_{k\to\infty} \boldsymbol{A}^k = 0$.

Proof. This is a standard result ■

Lemma 5 *If the condition* $(\mathfrak{C})$ *holds then* $\boldsymbol{A}$ *has spectral radius less than* 1.

Proof. Let $a_{ij}^{(l)}$ denote the entries for A^l and $r_i^{(l)}$ denote the sum of entries on the row i: $r_i^{(l)} = \sum_j a_{ij}^{(l)}$. For $l = 1$ we will omit the superscript. Note that if $l = p + q$ we have the following formula

$$r_i^{(l)} = \sum_j a_{ij}^{(p)} r_j^{(q)}.$$

More specifically, for $p = 1, q = l - 1$ we have

$$r_i^{(l)} = \sum_j a_{ij} r_j^{(l-1)}.$$

So by induction we have that $0 \le r_i^{(l)} \le 1$ for all i and l. Let $E \subset \{1, 2, \ldots n\}$ denote the set of nodes that have positive obligations to the external sector. So if $k \in E$ we have $r_k < 1$. Furthermore, for any $l \ge 1$

$$r_k^{(l)} = \sum_j a_{kj} r_j^{(l-1)} \le \sum_j a_{kj} = r_k < 1.$$

The condition that from every node i there exists a chain of positive obligations to some node k that has positive obligations to the external sector is equivalent to saying that for each i there exists a $k \in E$ and an m such that $a_{ik}^{(m)} > 0$. So for a large enough M we have

$$r_i^{(M)} = \sum_j a_{ij}^{(m)} r_j^{(M-m)} < \sum_j a_{ij}^{(m)} = r_i^{(m)} \le 1, \tag{45}$$

where the strict inequality above is due to the existence of an $j \in E$ such that $a_{ij}^{(m)} > 0$. It follows from (45) that the sum of entries on each row of $\boldsymbol{A}^M$ is strictly less than 1. Thus, from Lemma 3 we have $\rho(\boldsymbol{A}^M) < 1$ and so $\lim_{j\to\infty} \boldsymbol{A}^j = \lim_{k\to\infty} (\boldsymbol{A}^M)^k = 0$ where the last equality follows from Lemma 4. Applying Lemma 4 again we obtain that $\rho(\boldsymbol{A}) < 1$ ■

We are now ready to derive the uniqueness of the clearing payment vector, following Glasserman and Young (2015).

Lemma 6 *For any financial system $(\boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b})$ that satisfies condition $(\mathfrak{C})$ and a loss vector $\boldsymbol{x}$, there exists a unique clearing vector $\boldsymbol{p}(\boldsymbol{x})$.*

Proof. Note that

$$\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) : [\boldsymbol{0}, \overline{\boldsymbol{p}}] \to [\boldsymbol{0}, \overline{\boldsymbol{p}}]$$

is a contraction map since for any $\boldsymbol{p}$ and $\boldsymbol{p}'$, we have

$$\begin{aligned}
\|\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}) - \Phi(\boldsymbol{p}'; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})\| &= \left\|\min\{\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x}, \overline{\boldsymbol{p}}\} - \min\{\boldsymbol{A}^T\boldsymbol{p}' + \boldsymbol{c} - \boldsymbol{x}, \overline{\boldsymbol{p}}\}\right\| \\
&\le \left\|\boldsymbol{A}^T\boldsymbol{p} + \boldsymbol{c} - \boldsymbol{x} - (\boldsymbol{A}^T\boldsymbol{p}' + \boldsymbol{c} - \boldsymbol{x})\right\| \\
&= \left\|\boldsymbol{A}^T(\boldsymbol{p} - \boldsymbol{p}')\right\| \\
&\le \left\|\boldsymbol{A}^T\right\| \|\boldsymbol{p} - \boldsymbol{p}'\|
\end{aligned}$$

where $\left\|\boldsymbol{A}^T\right\| < 1$. Therefore by the Banach fixed-point theorem, $\Phi(\boldsymbol{p}; \boldsymbol{A}, \overline{\boldsymbol{p}}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x})$ has a unique fixed point ■