Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism

By Parag Pathak, Tayfun Sönmez
Harvard University

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Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism

Parag A. Pathak* Tayfun Sönmez†

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Abstract

Empirical and experimental evidence suggests different levels of sophistication among families in the Boston Public School student assignment plan. In this paper, we analyze the Nash equilibria of the preference revelation game induced by the Boston mechanism when there are two types of players. Sincere players are restricted to report their true preferences, while sophisticated players play a best response. We characterize the set of Nash equilibrium outcomes as the set of stable matchings of an economy with a modified priority structure, where sincere students lose their priority to sophisticated students. While there are multiple equilibrium outcomes, a sincere student receives the same assignment in all equilibria. Moreover any sophisticated student weakly prefers her assignment under the Pareto-dominant Nash equilibrium of the Boston mechanism to her assignment under the student-optimal stable mechanism, which was recently adopted by BPS for use starting with 2005-2006 school year.

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*Department of Economics, Harvard University, Cambridge MA 02138, ppathak@fas.harvard.edu

†Department of Economics, Boston College, Boston MA 02139, sonmezt@bc.edu

1 Introduction

In May 2005, Dr. Thomas Payzant, the Superintendent of Boston Public Schools (BPS), recommended to the public that the existing school choice mechanism in Boston (henceforth the **Boston mechanism**) should be replaced with an alternative mechanism that removes the incentives to "game the system" that handicapped the Boston mechanism.[1] The mechanism has been used by Boston to assign over 75,000 students to school from July 1999 until July 2005.[2] Following his recommendation, the Boston School Committee voted to replace the mechanism in July 2005 and adopt a new mechanism for the 2005-06 school year.

The major difficulty with the Boston mechanism is that students may benefit by submitting a rank order list that is different from their true underlying preferences over schools. Loosely speaking, the Boston mechanism attempts to assign as many students as possible to their first choice school, and only after all such assignments have been made does it consider assignments of students to their second choices, and so on. If a student is not admitted to her first choice school, her second choice may be filled with students who have listed it as their first choice. That is, a student may fail to get a place in her second choice school that would have been available had she listed that school as her first choice. If a student is willing to take a risk with her first choice, then she should be careful to rank a second choice that she has a chance of obtaining.

Some families understand these features of the Boston mechanism and have developed rules of thumb for how to submit preferences strategically. For instance, the West Zone Parents Group (WZPG), a well-informed group of approximately 180 members who meet regularly prior to admissions time to discuss Boston school choice for elementary school (grade K2), recommends two types of strategies to its members. Their introductory meeting minutes on 10/27/2003 state:

> One school choice strategy is to find a school you like that is undersubscribed and put it as a top choice, OR, find a school that you like that is popular and put it as a first choice and find a school that is less popular for a "safe" second choice.

Using data on stated choices from Boston Public Schools from 2000-2004, Abdulkadiroğlu, Pathak, Roth and Sönmez (2006) describe several empirical patterns which suggest that there

[1]The Boston mechanism is also widely used throughout several US school districts including Cambridge MA, Charlotte-Mecklenburg NC, Denver CO, Miami-Dade FL, Minneapolis MN, and Tampa-St. Petersburg FL.

[2]Between September 1989 and July 1999 thousands of students were assigned through another version of the same mechanism that imposed racial quotas. For the entire history of student assignment in Boston, see page 36 of the Student Assignment Task Force, submitted to Boston School Committee on September 22, 2004.

are different levels of sophistication among the families who participate in the mechanism. Some fraction of parents behave as the WZPG suggest and avoid ranking two overdemanded schools as their top two choices. On the other hand, nearly 20% of students list two overdemanded schools as their top two choices, and 27% of these students are unassigned by the mechanism.[3] This empirical evidence, together with the theoretical arguments in Abdulkadiroğlu and Sönmez (2003) and the experimental study of Chen and Sönmez (2006) was instrumental in the decision to replace the Boston mechanism with the **student-optimal stable mechanism** (Gale and Shapley 1962).

One of the remarkable properties of the student-optimal stable mechanism is that it is strategy-proof: truth-telling is a dominant strategy for each student. If families have access to advice on how to strategically modify their rank order lists from groups like the WZPG or through family resource centers, they can do no better than by submitting their true preferences to the mechanism. This feature was an important factor in Superintendent Payzant's recommendation to change the mechanism. The BPS Strategic Planning team, in their 05/11/2005 dated recommendation to implement a new BPS assignment algorithm, emphasized:[4]

> A strategy-proof algorithm "levels the playing field" by diminishing the harm done to parents who do not strategize or do not strategize well.

In this paper, we investigate the intuitive idea that replacing the Boston mechanism with the strategy-proof student-optimal stable mechanism "levels the playing field." To do so we consider a model with both sincere and sophisticated families,[5] analyze the Nash equilibria of the preference revelation game induced by the Boston mechanism (or simply the Nash equilibria of the **Boston game**), and compare the equilibrium outcomes with the dominant-strategy outcome of the student-optimal stable mechanism. In Proposition 1, we characterize the equilibrium outcomes of the Boston game as the set of stable matchings of a modified economy where sincere students lose their priorities to sophisticated students. This result implies that there exists a Nash equilibrium outcome where each student weakly prefers her assignment to any other equilibrium assignment. Hence, the Boston game is a coordination game among sophisticated students.

[3]When a student is unassigned by the mechanism, they are administratively assigned to a school that is not on their rank order list.

[4]See Recommendation to Implement a New BPS Algorithm - May 11, 2005, available online at http://boston.k12.ma.us/assignment/.

[5]This is also consistent with the experimental findings of Chen and Sönmez (2006) who have shown that about 20% of the subjects in the lab utilize the suboptimal strategy of truth-telling under the Boston mechanism.

We next examine properties of equilibria. While no sophisticated student loses priority to any other student, some of the sincere students may gain priority at a school at the expense of other sincere students by ranking the school higher on their preference list. As a result, a sincere student may still benefit from the Boston mechanism. In Proposition 2, we show that a sincere student receives the same assignment in all equilibria of the Boston game.

In Proposition 3, we compare the equilibria of the Boston game to the dominant-strategy outcome of the student-optimal stable mechanism. We show that any sophisticated student weakly prefers her assignment under the Pareto-dominant Nash equilibrium outcome of the Boston game over the dominant-strategy outcome of the student-optimal stable mechanism. When only some of the students are sophisticated, the Boston mechanism gives a clear advantage to sophisticated students provided that they can coordinate their strategies at a favorable equilibrium. This result might explain why, in testimony from the community about the Boston mechanism on 06/08/2005, the leader of the WZPG opposed changing the mechanism:

> There are obviously issues with the current system. If you get a low lottery number and don't strategize or don't do it well, then you are penalized. But this can be easily fixed. When you go to register after you show you are a resident, you go to table B and the person looks at your choices and lets you know if you are choosing a risky strategy or how to re-order it.
>
> Don't change the algorithm, but give us more resources so that parents can make an informed choice.

The position of the WZPG may be interpreted as a desire to maintain their strategic advantage over sincere students under the Boston mechanism. In a model where all students are sophisticated, the set of Nash equilibrium outcomes of the Boston game coincides with the set of stable matchings of the underlying economy (Ergin and Sönmez 2006).[6] This theoretical result would suggest that a transition to the student-optimal stable mechanism will be embraced by all student groups for it would be in the best interest of all students. In contrast, Proposition 3 may explain why the WZPG did not embrace the transition.

Our last result, Proposition 4, examines what happens when a sincere student becomes sophisticated. Comparing the Pareto-dominant Nash equilibrium outcomes of the two scenarios, the student in question is weakly better off when she is sophisticated although each other sophisticated student weakly prefers that she remained sincere.

[6] Kojima (2006) extends this result to a model with substitutable priorities (Kelso and Crawford 1981).

The layout of the paper is as follows. Section 2 defines the model and Section 3 characterizes the set of equilibrium. Section 4 presents comparative statics and Section 5 concludes. Finally the Appendix contains the proofs.

2 The Model

In a **school choice problem** (Abdulkadiroğlu and Sönmez 2003) there are a number of students each of whom should be assigned a seat at one of a number of schools. Each student has a strict preference ordering over all schools as well as remaining unassigned and each school has a strict priority ranking of all students. Each school has a maximum capacity.

Formally, a school choice problem consists of:

1. a set of students $I = \{i_1, ..., i_n\}$,
2. a set of schools $S = \{s_1, ..., s_m\}$,
3. a capacity vector $q = (q_{s_1}, ..., q_{s_m})$,
4. a list of strict student preferences $P_I = (P_{i_1}, ..., P_{i_n})$, and
5. a list of strict school priorities $\pi = (\pi_{s_1}, ..., \pi_{s_m})$.

For any student i, P_i is a strict preference relation over $S \cup \{i\}$ where sP_ii means student i strictly prefers a seat at school s to being unassigned. For any student i, let R_i donote the "at least as good as" relation induced by P_i. For any school s, the function $\pi_s : \{1, \ldots, n\} \to \{i_1, \ldots, i_n\}$ is the priority ordering at school s where $\pi_s(1)$ indicates the student with highest priority, $\pi_s(2)$ indicates the student with second highest priority, and so on. Priority rankings are determined by the school district and schools have no control over them. We fix the set of students, the set of schools and the capacity vector throughout the paper; hence the pair (P, π) denotes a school choice problem (or simply an **economy**).

The school choice problem is closely related to the well-known college admissions problem (Gale and Shapley 1962). The main difference is that in college admissions each school is a (possibly strategic) agent whose welfare matters, whereas in school choice each school is a collection of indivisible goods to be allocated and only the welfare of students is considered.

The outcome of a school choice problem, as in college admissions, is a **matching**. Formally a matching $\mu : I \to S \cup I$ is a function such that

1. $\mu(i) \notin S \Rightarrow \mu(i) = i \qquad$ for any student i, and

2. $|\mu^{-1}(s)| \leq q_s$ for any school s.

We refer $\mu(i)$ as the assignment of student i under matching μ.

A matching μ **Pareto dominates** (or is a **Pareto improvement** over) a matching ν, if $\mu(i) R_i \nu(i)$ for all $i \in I$ and $\mu(i) P_i \nu(i)$ for some $i \in I$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

A mechanism is a systematic procedure that selects a matching for each economy.

2.1 The Boston Student Assignment Mechanism

The Boston mechanism is by far the most popular mechanism that is used in school districts throughout the U.S. For any economy, the outcome of the Boston mechanism is determined in several rounds with the following procedure:

> *Round 1*: In Round 1, only the first choices of students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until there are no seats left or there is no student left who has listed it as their first choice.
>
> In general, at
>
> *Round k*: Consider the remaining students. In Round k, only the k^{th} choices of these students are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice and assign the remaining seats to these students one at a time following their priority order until there are no seats left or there is no student left who has listed it as his k^{th} choice.
>
> The procedure terminates when each student is assigned a seat at a school.

The Boston mechanism induces a preference revelation game among students. We refer to this game as the **Boston game**.

2.2 Sincere and Sophisticated Students

We assume that there are two types of students: sincere and sophisticated. Let N, M denote sets of sincere and sophisticated, respectively. We have $N \cup M = I$ and $N \cap M = \emptyset$. Sincere students are unaware about the strategic aspects of the student assignment process and they simply reveal their preferences truthfully. The strategy space of each sincere student is a singleton under the Boston game. Each sophisticated student, on the other hand, recognizes the strategic aspects of the student assignment process, and the support of her strategy space is all strict preferences over the set of schools plus remaining unassigned. We focus on the Nash equilibria of the Boston game where only sophisticated students are active players. Each sophisticated student selects a best response to the other students.

2.3 Stability

The following concept, which plays a central role in the analysis of two-sided matching markets, will be useful to characterize the Nash equilibria of the Boston game.

A matching μ is **stable** if

1. it is **individually rational** in the sense that there is no student i who prefers remaining unassigned to her assignments $\mu(i)$, and
2. there is no student-school pair (i, s) such that,
   (a) student i prefers s to her assignment $\mu(i)$, and
   (b) either school s has a vacant seat under μ or there is a lower priority student j who nonetheless received a seat at school s under μ.

Gale and Shapley (1962) show that the set of stable matchings is non-empty and there exists a stable matching, the **student-optimal stable matching**, that each student weakly prefers to any other stable matching. We refer the mechanism that selects this stable matching for each problem as the **student-optimal stable mechanism**. Dubins and Freedman (1981) and Roth (1982) show that under the student-optimal stable mechanism for each student, truth-telling is a dominant strategy.

2.4 An Illustrative Example

Since a student "loses" her priority to students who rank a school higher in their rank order list, the outcome of the Boston mechanism is not necessarily stable. However, Ergin and

Sönmez (2006) show that any Nash equilibrium outcome of the Boston game is stable when all students are sophisticated. Based on this result they have argued that a change from the Boston mechanism to the student-optimal stable mechanism should be embraced by all students for it will result in a Pareto improvement. This is not what happened in summer 2005 when Boston Public Schools gave up the Boston mechanism and adopted the student-optimal stable mechanism. A simple example provides some insight on the resistance of sophisticated players to the change of the mechanism.

Example 1. There are three schools a, b, c each with one seat and three students i_1, i_2, i_3. The priority list $\pi = (\pi_a, \pi_b, \pi_c)$ and student utilities representing their preferences $P = (P_{i_1}, P_{i_2}, P_{i_3})$ are as follows:

	a	b	c
u_{i_1}	1	2	0
u_{i_2}	0	2	1
u_{i_3}	2	1	0

$$\begin{array}{ll} \pi_a: & i_2 - i_1 - i_3 \\ \pi_b: & i_3 - i_2 - i_1 \\ \pi_c: & i_2 - i_3 - i_1 \end{array}$$

Students i_1 and i_2 are sophisticated whereas student i_3 is sincere. Hence the strategy space of each of students i_1, i_2 is $\{abc, acb, bac, bca, cab, cba\}$ whereas the strategy space of student i_3 is the singleton $\{abc\}$. We have the following $6\times 6\times 1$ Boston game for this simple example:

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	(0,0,1)	(0,0,1)	**(1,2,0)**	**(1,2,0)**	(1,1,1)	(1,1,1)
acb	(0,0,1)	(0,0,1)	**(1,2,0)**	**(1,2,0)**	(1,1,1)	(1,1,1)
bac	(2,0,0)	(2,0,0)	(0,2,2)	(0,2,2)	(2,1,2)	(2,1,2)
bca	(2,0,0)	(2,0,0)	(0,2,2)	(0,2,2)	(2,1,2)	(2,1,2)
cab	(0,0,1)	(0,0,1)	(0,2,2)	(0,2,2)	(0,2,2)	(0,2,2)
cba	(0,0,1)	(0,0,1)	(0,2,2)	(0,2,2)	(0,2,2)	(0,2,2)

where the row player is student i_1 and the column player is student i_2.

There are four Nash equilibrium profiles of the Boston game (indicated in boldface) each with a Nash equilibrium payoff of (1,2,0) and a Nash equilibrium outcome of

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & c \end{pmatrix}.$$

We have the following useful observations about the equilibria:

1. Truth-telling, i.e. the profile (bac, bca, abc), is not a Nash equilibrium of the Boston game.

2. Unlike in Ergin and Sönmez (2006), the Nash equilibrium outcome μ is not a stable matching of the economy (P, π). The sincere student i_3 not only prefers school b to her assignment $\mu(i_3) = c$ but also she has the highest priority there. Nevertheless, by being truthful and ranking b second, she has lost her priority to student i_2 at equilibria.

3. The unique stable matching of the economy (P, π) is

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix}.$$

   Matchings μ and ν are not Pareto ranked. While the sophisticated student i_1 is indifferent between the two matchings, the sophisticated student i_2 is better off under matching μ and the sincere student i_3 is better off under matching ν. That is, the sophisticated student i_2 is better off under the Nash equilibria of the Boston game at the expense of the sincere student i_3.

We next characterize the Nash equilibrium outcomes of the Boston game which will be useful to generalize the above observations.

3 Characterization of Nash Equilibrium Outcomes

Given an economy (P, π), we will construct an augmented economy that will be instrumental in describing the set of Nash equilibrium outcomes of the Boston game.

Given an economy (P, π) and a school s, partition the set of students I into m sets as follows:

I_1^s: Sophisticated students and sincere students who rank s as their first choices under P,

I_2^s: sincere students who rank s as their second choices under P,

I_3^s: sincere students who rank s as their third choices under P,

$\vdots$ $\qquad\qquad\qquad\qquad \vdots$

I_m^s: sincere students who rank s as their last choices under P.

Given an economy (P, π) and a school s, construct an **augmented priority ordering** $\tilde{\pi}_s$ as follows:

- each student in I_1^s has higher priority than each student in I_2^s, each student in I_2^s has higher priority than each student in I_3^s, ..., each student in I_{m-1}^s has higher priority than each student in I_m^s, and
- for any $k \leq m$, priority among students in I_k^s is based on π_s.

Define $\tilde{\pi} = (\tilde{\pi}_s)_{s \in S}$. We refer the economy $(P, \tilde{\pi})$ as the **augmented economy**.

Example 1 continued. Let us construct the augmented economy for Example 1. Since only student i_3 is sincere, $\tilde{\pi}$ is constructed from π by pushing student i_3 to the end of the priority ordering at each school except her top choice a (where she has the lowest priority to begin with):

$$\begin{aligned}
\pi_a : i_2 - i_1 - i_3 \quad &\Rightarrow \quad \tilde{\pi}_a : i_2 - i_1 - i_3 \\
\pi_b : i_3 - i_2 - i_1 \quad &\Rightarrow \quad \tilde{\pi}_b : i_2 - i_1 - i_3 \\
\pi_c : i_1 - i_3 - i_2 \quad &\Rightarrow \quad \tilde{\pi}_c : i_1 - i_2 - i_3
\end{aligned}$$

The key observation is that the unique Nash equilibrium outcome μ of the Boston game is the unique stable matching for the augmented economy $(P, \tilde{\pi})$.

While the uniqueness is specific to the above example, the equivalence is general. We are ready to present our first result.

Proposition 1: The set of Nash equilibrium outcomes of the Boston game under (P, π) is equivalent to the set of stable matchings under $(P, \tilde{\pi})$.

Therefore at Nash equilibria sophisticated students gain priority at the expense of sincere students. Another implication of Proposition 1 is that the set of equilibrium outcomes inherits some of the properties of the set of stable matchings. In particular there is a Nash equilibrium outcome of the Boston game that is weakly preferred to any other Nash equilibrium outcome by all students. We refer this outcome as the **Pareto-dominant Nash equilibrium outcome**. Hence the Boston game is a coordination game among sophisticated students.

Equilibrium Assignments of Sincere Students

The student-optimal stable mechanism replaced the Boston mechanism in Boston in 2005. In the following section we will compare the equilibrium outcomes of the Boston game with the dominant-strategy equilibrium outcome of the student-optimal stable mechanism. One of the difficulties in such comparative static analysis is that the Boston game has multiple equilibria in general. Nevertheless, as we present next, multiplicity is not an issue for sincere students.

Proposition 2: Let μ, ν be both Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism. For any sincere student $i \in N$, $\mu(i) = \nu(i)$.

While sincere students are only passive players under the Boston game, their outcome depends on the strategy choices of all students. Nevertheless, as we present in Proposition 2, equilibrium choice in Boston game has no bite on the assignment of a sincere student. Although her assignment depends on the strategy choices of the sophisticated students, it does not depend on which equilibrium strategy is played.[7]

4 Comparative Statics

4.1 Comparing Mechanisms

The outcome of the student-optimal stable mechanism can be obtained with the following **student-proposing deferred acceptance algorithm** (Gale and Shapley 1962):

> *Step 1*: Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.
>
> In general, at:
>
> *Step k*: Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

[7] Proposition 2 does not require that sincere students report their true preferences to the mechanism. The same result is true when sincere students play any fixed strategy.

The algorithm terminates when no student proposal is rejected and each student is assigned her final tentative assignment. Any student who is not holding a tentative assignment remains unassigned.

4.1.1 Comparing Mechanisms for Sincere Students

Sincere students lose priority to sophisticated students under the Boston mechanism. They may also be affected by other sincere students, so that some sincere students may benefit at the expense of other sincere students under the Boston mechanism. More precisely, a sincere student may prefer the Boston mechanism to the student-optimal stable mechanism since:

- she gains priority at her first choice school over sincere students who rank it second or lower, and in general
- she gains priority at her k^{th} choice school over sincere students who rank it $(k+1)^{\text{th}}$ or lower, etc.

Example 2. There are three schools a, b, c each with one seat and three sincere students i_1, i_2, i_3. Preferences and priorities are as follows:

$$\begin{array}{lll} P_{i_1}: & a\ b\ c & \qquad \pi_a: & i_1 - i_2 - i_3 \\ P_{i_2}: & a\ b\ c & \qquad \pi_b: & i_2 - i_2 - i_3 \\ P_{i_3}: & b\ a\ c & \qquad \pi_c: & i_1 - i_2 - i_3 \end{array}$$

Outcomes of the Boston mechanism and the student-optimal stable mechanism are

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & c \end{pmatrix},$$

respectively. Under the Boston mechanism the sincere student i_3 gains priority at her top choice school b over the sincere student i_2. Hence student i_3 prefers her assignment under the Boston mechanism whereas student i_2 prefers her assignment under the student-optimal stable mechanism.

4.1.2 Comparing Mechanisms for Sophisticated Students

Unlike a sincere student, a sophisticated student may be assigned seats at different schools at different equilibrium outcomes of the Boston game. Hence we will concentrate on the Pareto-dominant Nash equilibrium outcome of the Boston game.

Proposition 3: The school a sophisticated student receives in the Pareto-dominant equilibrium of the Boston mechanism is weakly better than her dominant-strategy outcome under the student-optimal stable mechanism.

While the Boston mechanism is easy to describe, it induces a complicated coordination game among sophisticated students. Therefore, it is important to be cautious in interpreting Proposition 3. The result relies on sophisticated students and their families being able to reach the Pareto-dominant Nash equilibrium outcome. In a school district where this is a good approximation, sophisticated students may prefer keeping the Boston mechanism in order to capitilize on their strategic advantage.

4.2 Becoming Sophisticated

Our final result concerns a sincere student i who becomes sophisticated. While student i weakly benefits from this transition under the Pareto-dominant Nash equilibrium of the Boston game, students who have been sophisticated weakly suffer.

To state this result, we must define additional notation. First, fix an economy (P, π). Let $M_1 \subset I$ be the set of sophisticated students and N_1 be the set of sincere students. Next consider an initially sincere student $i \in N_1$ and suppose she becomes sophisticated. Let $M_2 = M_1 \cup \{i\}$ be the set of sophisticated students including i, and let $N_2 = N_1 \setminus \{i\}$ be the set of remaining sincere students.

Let ν^I be the Pareto-dominant Nash equilibrium of the Boston game where M_1 and N_1 are the sophisticated and sincere players, respectively. Let μ^I be the Pareto-dominant Nash equilibrium of the Boston game where M_2 and N_2 are the sophisticated and sincere players respectively.

Proposition 4: Let i, M_1, ν^I, μ^I be as described above. Student i weakly benefits from becoming sophisticated in the Pareto-dominant Nash equilibrium of the Boston game, whereas all other sophisticated students weakly suffer. That is,

$$\mu^I(i) \, R_i \, \nu^I(i) \quad \text{and} \quad \nu^I(j) \, R_j \, \mu^I(j) \quad \text{for all } j \in M_1.$$

This proposition suggests that groups such as the West Zone Parents Group do not exist only to share information on how to become strategic because educating a sincere player will not benefit an existing sophisticated player. Rather, this proposition suggests that the theoretical function of the West Zone Parents Group may be to coordinate behavior among the sophisticated players.

5 Conclusion

Boston Public Schools stated that their main rationale for changing their student assignment system is that it levels the playing field. They identified a fairness rationale for a strategy-proof system. In this paper, we examined this intuitive notion and showed that the Boston mechanism favors strategic parents at Pareto-dominant Nash equilibrium, providing formal support for BPS's position.

Despite its theoretical weaknesses, poor performance in laboratory experiments, and empirical evidence of suboptimal play, the Boston mechanism is the most widely used school choice mechanism in the United States. This paper proposes another theoretical rationale for abandoning the mechanism based on fairness or equal access, which was central in Boston's decision.

It is remarkable that such a flawed mechanism is so widely used throughout school districts. Chubb and Moe (1999) argue that important stakeholders often control the mechanisms of reform in education policy. In the context of student assignment mechanisms, the important stakeholders may be sophisticated parents who have invested energy in learning about the mechanism, and the choice of the Boston mechanism may reflect their preferences.

Appendix: Proofs

Proof of Proposition 1:

$\Leftarrow$ (Any stable matching under $(P, \tilde{\pi})$ is an equilibrium outcome of the Boston game under (P, π)):

Fix an economy (P, π) and let μ be stable under $(P, \tilde{\pi})$. Let preference profile Q be such that $Q_i = P_i$ for all $i \in N$ and $\mu(i)$ is the first choice under Q_i for all $i \in M$. Matching μ is stable under $(Q, \tilde{\pi})$ as well. Let ν be the outcome of the Boston mechanism under (Q, π). We first show, by induction, that $\nu = \mu$.

Consider any student j who does not receive her first choice s_j^1 under Q at matching μ. By construction of Q, student j is naive. Since μ is stable under $(Q, \tilde{\pi})$ and since student j does not lose priority to any student at school s_j^1 when priorities change from π to $\tilde{\pi}$, she has lower priority under $\pi_{s_j^1}$ than any student who has received a seat at s_j^1 under μ. Each of these students rank s_j^1 as their first choices under Q and school s_j^1 does not have empty seats under μ for otherwise (j, s) would block μ under $(Q, \tilde{\pi})$. Therefore $\nu(j) \neq s_j^1$. So a student can receive her first choice under Q at matching ν only if she receives her first choice under Q at matching μ. But then, since the Boston mechanism is Pareto efficient, matching ν is Pareto efficient under (Q, π) which in turn implies that $\nu(i) = \mu(i)$ for any student i who receives her first choice under Q at matching μ.

Next given $k > 1$, suppose

1. any student who does not receive one of her top k choices under Q at matching μ does not receive one of her top k choices under Q at matching ν either, and

2. for any student i who receives one of her top k choices under Q at matching μ, $\nu(i) = \mu(i)$.

We will show that the same holds for $(k+1)$ and this will establish that $\nu = \mu$. Consider any student j who does not receive one of her top $k+1$ choices under Q at matching μ. By construction of Q, student j is naive and by assumption she does not receive one of her top k choices under Q at matching ν. Consider $(k+1)^{\text{th}}$ choice s_j^{k+1} of student j under Q_j. Since μ is stable under $(Q, \tilde{\pi})$, there is no empty seat at school s_j^{k+1} for otherwise pair (j, s_j^{k+1}) would block matching μ under $(Q, \tilde{\pi})$. Moreover since μ is stable under $(Q, \tilde{\pi})$, for any student i with $\mu(i) = s_j^{k+1}$ one of the following three cases should hold:

1. $i \in M$ and by construction s_j^{k+1} is her first choice under Q_i,

2. $i \in N$ and s_j^{k+1} is one of her top k choices under Q_i,

3. $i \in N$, she has ranked s_j^{k+1} as her $(k+1)^{\text{th}}$ choice under Q_i, and she has higher priority than j under $\pi_{s_j^{k+1}}$.

If either of the first two cases holds, then $\nu(i) = s_j^{k+1}$ by inductive assumption. If Case 3 holds, then student i has not received one of her top k choices under Q_i at matching ν by the inductive assumption and furthermore she has ranked school s_j^{k+1} as her $(k+1)^{\text{th}}$ choice under Q_i. Since she has higher priority than j under $\pi_{s_j^{k+1}}$, $\nu(j) = s_j^{k+1}$ implies $\nu(i) = s_j^{k+1}$. Therefore considering all three cases, $\nu(j) = s_j^{k+1}$ implies $\nu(i) = s_j^{k+1}$ for any student i with $\mu(i) = s_j^{k+1}$ and since school s_j^{k+1} does not have empty seats under μ, $\nu(j) \neq s_j^{k+1}$. So a student can receive one of her top $k+1$ choices under Q at matching ν only if she receives one of her top $k+1$ choices under Q at matching μ. Moreover matching ν is Pareto efficient under (Q, π) and therefore $\nu(i) = \mu(i)$ for any student i who receives her $(k+1)^{\text{th}}$ choice under Q_i at μ completing the induction and establishing $\nu = \mu$.

Next we show that Q is a Nash equilibrium profile and hence ν is a Nash equilibrium outcome. Consider any sophisticated student $i \in M$ and suppose $s P_i \nu(i) = \mu(i)$ for some school $s \in S$. Since $\nu = \mu$ is stable under $(Q, \tilde{\pi})$ and since student i gains priority under $\tilde{\pi}_s$ over only students who rank s second or worse under Q, not only any student $j \in I$ with $\nu(j) = s$ ranks school s as her first choice under Q_i but she also has higher priority under π_s. Therefore regardless of what preferences student i submits, each student $j \in I$ with $\nu(j) = s$ will receive a seat at school s. Moreover by stability of $\nu = \mu$ under $(Q, \tilde{\pi})$ there are no empty seats at school s and hence student i cannot receive a seat at s regardless of her submitted preferences. Therefore matching ν is a Nash equilibrium outcome.

$\Rightarrow$ (Any equilibrium outcome of the Boston game under (P, π) is a stable matching under $(P, \tilde{\pi})$):

Suppose matching μ is not stable under $(P, \tilde{\pi})$. Let Q be any preference profile where $Q_i = P_i$ for any naive student i and where μ is the outcome of the Boston mechanism under (Q, π). We will show that Q is not a Nash equilibrium strategy profile of the Boston game under (P, π).

First suppose μ is not individually rational under $(P, \tilde{\pi})$. Then there is a student $i \in I$ with $iP_i\mu(i)$. Since the Boston mechanism is individually rational, student i should be a sophisticated student who has ranked the unacceptable school $\mu(i)$ as acceptable. Let P_i^0 be a preference relation where there is no acceptable school. Upon submitting P_i^0, student i will profit by getting unassigned. Hence Q cannot be an equilibrium profile in this case.

Next suppose there is a pair (i, s) that blocks μ under $(P, \tilde{\pi})$. Since μ is the outcome of the Boston mechanism under (Q, π), student i cannot be a naive student. Let P_i^s be a preference relation where school s is the first choice. We have two cases to consider:

Case 1: School s has an empty seat at μ.

Recall that by assumption μ is the outcome of the Boston mechanism under (Q, π). Since s has an empty seat at μ, there are fewer students who rank s as their first choice under Q than the capacity of school s. Therefore upon submitting the preference relation P_i^s, student i will profit by getting assigned a seat at school s. Hence Q cannot be an equilibrium profile.

Case 2: School s does not have an empty seat at μ.

By assumption μ is the outcome of the Boston mechanism under (Q, π) and there is a student j with $\mu(j) = s$ although i has higher priority than j under $\tilde{\pi}_s$. If school s is not j's first choice under Q_j then there are fewer students who rank s as their first choice under Q than the capacity of school s, and upon submitting the preference relation P_i^s, student i will profit by getting assigned a seat at school s contradicting Q being an equilibrium profile. If on the other hand school s is j's first choice under Q_j, then either j is sophisticated or j is naive and s is her first choice under P_j. In either case i having higher priority than j under $\tilde{\pi}_s$ implies i having higher priority than j under π_s. Moreover since $\mu(j) = s$, the capacity of school s is strictly larger than the number of students who both rank it as their first choice under Q and also has higher priority than j under π_s. Therefore the capacity of school s is strictly larger than the number of students who both rank it as their first choice under Q and also has higher priority than i under π_s. Hence upon submitting the preference relation P_i^s, student i will profit by getting assigned a seat at school s contradicting Q being an equilibrium profile.

Since there is no Nash equilibrium profile Q for which μ is the outcome of the Boston mechanism under (Q, π), μ cannot be a Nash equilibrium outcome of the Boston game under (P, π). $\square$

Proof of Proposition 2: Fix an economy (P, π). Let μ, ν be both Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism. By Proposition 1, μ, ν are stable matchings under $(P, \tilde{\pi})$. Let $\overline{\mu} = \mu \vee \nu$ and $\underline{\mu} = \mu \wedge \nu$ be the join and meet of the stable matching lattice. That is, $\overline{\mu}, \underline{\mu}$ are such that, for all $i \in I$,

$$\overline{\mu}(i) = \begin{cases} \mu(i) & \text{if } \mu(i) R_i \nu(i) \\ \nu(i) & \text{if } \nu(i) R_i \mu(i) \end{cases} \qquad \underline{\mu}(i) = \begin{cases} \nu(i) & \text{if } \mu(i) R_i \nu(i) \\ \mu(i) & \text{if } \nu(i) R_i \mu(i) \end{cases}$$

Since the set of stable matchings is lattice (attributed to John Conway by Knuth 1976), $\overline{\mu}$ and $\underline{\mu}$ are both stable matchings under $(P, \tilde{\pi})$.

Let $T = \{i \in I : \overline{\mu}(i) \neq \underline{\mu}(i)\}$. That is, T is the set of students who receive a different assignment under $\overline{\mu}$ and $\underline{\mu}$. If $T \subseteq M$, then we are done. So suppose there exists $i \in T \cap N$. We will show that this leads to a contradiction. Let $s = \overline{\mu}(i)$, $s^* = \underline{\mu}(i)$, and $j \in \overline{\mu}^{-1}(s^*) \cap T$. Such a student $j \in I$ exists because by the rural hospitals theorem of Roth (1985) the same set of students and the same set of seats are assigned under any pair of stable matchings. Note that $j \in \overline{\mu}^{-1}(\underline{\mu}(i))$.

Claim: $j \in N$.

Proof of the Claim: By construction of $\overline{\mu}$ and $\underline{\mu}$, $s P_i s^*$ and therefore school s^* is not i's first choice. Moreover by Roth and Sotomayor (1989) each student in $\underline{\mu}(s^*) \backslash \overline{\mu}(s^*)$ has higher priority under $\tilde{\pi}_{s^*}$ than each student in $\overline{\mu}(s^*) \setminus \underline{\mu}(s^*)$, and hence i has higher priority than j under $\tilde{\pi}_{s^*}$. But since i is naive by assumption and since s^* is not her first choice, student j has to be naive as well for otherwise she would have higher priority under $\tilde{\pi}_{s^*}$. $\diamondsuit$

Next construct the following directed graph: Each student $i \in T \cap N$ is a node and there is a directed link from $i \in T \cap N$ to $j \in T \cap N$ if $j \in \overline{\mu}^{-1}(\underline{\mu}(i))$. By the above Claim there is at least one directed link emanating from each node. Therefore, since there are finite number of nodes, there is at least one cycle in this graph. Pick any such cycle. Let $T_1 \subseteq T \cap N$ be the set of students in the cycle, and let $|T_1| = k$. Relabel students in T_1 and their assignments under $\overline{\mu}$, $\underline{\mu}$ so that the restriction of matchings $\overline{\mu}$ and $\underline{\mu}$ to students in T_1 is as follows:

$$\overline{\mu}_{T_1} = \begin{pmatrix} i^1 & i^2 & ... & i^k \\ s^1 & s^2 & ... & s^k \end{pmatrix} \qquad \underline{\mu}_{T_1} = \begin{pmatrix} i^1 & i^2 & ... & i^{k-1} & i^k \\ s^2 & s^3 & ... & s^k & s^1 \end{pmatrix}$$

Note that a school may appear more than once in a cycle so that schools s^t, s^u does not need to be distinct for $t \neq u$ (although they would have if the cycle we pick is minimal). This has no relevance for the contradiction we present next.

Let $r_{i,s}$ be the ranking of school s in P_i (so $r_{i,s} = \ell$ means that s is i's ℓth choice). By Roth and Sotomayor (1989) i^k has higher priority at school s^1 than i^1 under $\tilde{\pi}_{s^1}$, and since i^1, i^k are both naive,

$$r_{i^k,s^1} \leq r_{i^1,s^1}$$

Similarly

$$\begin{gathered} r_{i^1,s^2} \leq r_{i^2,s^2} \\ \vdots \\ r_{i^{k-1},s^k} \leq r_{i^k,s^k} \end{gathered}$$

Moreover since $\overline{\mu}(i) P_i \underline{\mu}(i)$ for each $i \in T$,

$$\begin{array}{lcl} s^1 P_{i^1} s^2 & \Rightarrow & r_{i^1,s^1} < r_{i^1,s^2} \\ s^2 P_{i^2} s^3 & \Rightarrow & r_{i^2,s^2} < r_{i^2,s^3} \\ & \vdots & \\ s^{k-1} P_{i^{k-1}} s^k & \Rightarrow & r_{i^{k-1},s^{k-1}} < r_{i^{k-1},s^k} \\ s^k P_{i^k} s^1 & \Rightarrow & r_{i^k,s^k} < r_{i^k,s^1} \end{array}$$

Combining the inequalities, we obtain

$$r_{i^k,s_1} \leq r_{i^1,s^1} < r_{i^1,s^2} \leq r_{i^2,s^2} < r_{i^2,s^3} \leq ... \leq r_{i^{k-1},s^{k-1}} < r_{i^{k-1},s^k} \leq r_{i^k,s^k} < r_{i^k,s^1}$$

establishing the desired contradiction. Hence there exists no $i \in N$ with $\overline{\mu}(i) \neq \underline{\mu}(i)$. But that means there exists no $i \in N$ with $\mu(i) \neq \nu(i)$ completing the proof. $\square$

The following lemma will be useful to prove Proposition 3 and Proposition 4. We need the following piece of notation to present this lemma. Given a preference profile P and a school s, let $F_s(P)$ denote the set of students who rank school s as their first choice under P.

Lemma 1: Fix a preference profile P, a list of priorities π, and a set of students $J \subset I$. Let priorities σ be such that, for any school s:

1. any student in $J \cup F_s(P)$ has higher priority under σ_s than any student in $I \setminus (J \cup F_s(P))$, and
2. for any student $j \in J \cup F_s(P)$ and any student $i \in I$, if j has higher priority than i under π_s then j also has higher priority than i under σ_s.

Let μ^I, ν^I be the student-optimal stable matching for economies (P, π), (P, σ) respectively. Then:

$$\nu^I(j) \, R_j \, \mu^I(j) \qquad \text{for any } j \in J.$$

Proof of Lemma 1: Fix P and $J \subset I$ and let priorities π, σ be as in the statement of the lemma. Let μ^I, ν^I be the student-optimal stable matching for economies (P, π), (P, σ) respectively. Define matching ν_0 as follows:

$$\begin{aligned} \nu_0(j) &= \mu^I(j) \qquad \text{for all } j \in J, \\ \nu_0(i) &= i \qquad \text{for all } i \in I \setminus J. \end{aligned}$$

If ν_0 is stable under (P, σ), then each student $i \in J$ weakly prefers $\nu^I(j)$ to $\nu_0(j) = \mu^I(j)$ and we are done. So w.l.o.g. assume ν_0 is not stable under (P, σ). We will construct a sequence of matchings $\nu_0, \nu_1, \ldots, \nu_k$ where ν_k is stable under (P, σ), and

$$\nu_\ell(j) \, R_j \, \nu_{\ell-1}(j) \qquad \text{for all } j \in J \text{ and } \ell \geq 1.$$

Consider matching ν_0. Since ν_0 is not stable but individually rational under (P, σ), there is a blocking pair. Pick any school s^1 in a blocking pair and let i^1 be the highest priority student under σ_{s^1} who strictly prefers s^1 to her assignment under ν_0.

Claim 1: School s^1 has an empty seat under ν_0.

Proof of Claim 1: We have three cases to consider.

Case 1: $i^1 \in J$.

By construction $\nu_0(i^1) = \mu^I(i^1)$, $\nu_0^{-1}(s^1) = (\mu^I)^{-1}(s^1) \cap J$ and (i^1, s^1) does not block μ^I under (P, π). When priorities change from π to σ, no student in J loses priority to student i^1 and therefore (i^1, s^1) can block ν_0 under (P, σ) only if school s^1 has an empty seat under ν_0.

Case 2: $i^1 \in I \setminus J$ and $i^1 \notin F_{s^1}(P)$.

Student i^1 has lower priority under σ_{s^1} than any student in J. Since $\nu_0^{-1}(s^1) \subseteq J$, pair (i^1, s^1) can block ν_0 under (P, σ) only if school s^1 has an empty seat under ν_0.

Case 3: $i^1 \in I \setminus J$ and $i^1 \in F_{s^1}(P)$.

If $\mu^I(i^1) = s^1$, then by construction the seat i^1 occupies at s^1 under μ^I is empty under ν_0. If $\mu^I(i^1) \neq s^1$, then i^1 is assigned a seat at a less preferred school and hence all seats at s^1 are occupied under μ^I by higher priority students under π_{s^1}. Since (i^1, s^1) blocks ν^0 under (P, σ), at least one of these students must be a student in $I \setminus (J \cup F_{s^1}(P))$. That is because student i^1 gains priority over only these student under σ_{s^1}. Since $I \setminus (J \cup F_{s^1}(P)) \subseteq I \setminus J$, at least one seat at s^1 must be empty under ν_0.

This completes the proof of Claim 1. $\diamondsuit$

Construct matching ν_1 by satisfying pair (i^1, s^1) at matching ν_0:

$$\begin{aligned} \nu_1(i) &= \nu_0(i) \qquad \text{for all } i \in I \setminus \{i^1\}, \\ \nu_1(i^1) &= s^1 \end{aligned}$$

By construction $\nu_1(i) \, R_i \, \nu_0(i)$ for all $i \in I$. If ν_1 is stable under (P, σ), then for all $j \in J$,

$$\nu^I(j) \, R_j \, \nu_1(j) \, R_j \, \underbrace{\nu_0(j)}_{=\mu^I(j)}$$

and we are done. If not, we proceed with the construction of matching ν_2.

In general for any $\ell > 0$, if ν_ℓ is not stable under (P, σ) construct $\nu_{\ell+1}$ as follows: Pick any school $s^{\ell+1}$ in a blocking pair for ν_ℓ and let $i^{\ell+1}$ be the highest priority student under $\sigma_{s^{\ell+1}}$ who strictly prefers $s^{\ell+1}$ to her assignment under ν_ℓ. As we prove next, school $s^{\ell+1}$ has an empty seat under ν_ℓ. Construct matching $\nu_{\ell+1}$ by satisfying pair $(i^{\ell+1}, s^{\ell+1})$ at matching ν_ℓ:

$$\begin{aligned} \nu_{\ell+1}(i) &= \nu_\ell(i) \qquad \text{for all } i \in I \setminus \{i^{\ell+1}\}, \\ \nu_{\ell+1}(i^{\ell+1}) &= s^{\ell+1} \end{aligned}$$

Claim 2: School $s^{\ell+1}$ has an empty seat under ν_ℓ.

Proof of Claim 2: We have three cases to consider.

Case 1: $i^{\ell+1} \in J$.

By construction

$$\nu_\ell(i^{\ell+1})\, R_{i^{\ell+1}}\, \underbrace{\nu_0(i^{\ell+1})}_{=\mu^I(i^{\ell+1})} \quad \text{and} \quad \nu_\ell^{-1}(s^{\ell+1}) \subseteq [(\mu^I)^{-1}(s^{\ell+1}) \cap J] \cup \{i^1, \ldots, i^\ell\}.$$

Since $(i^{\ell+1}, s^{\ell+1})$ does not block μ^I under (P, π) and since student $i^{\ell+1}$ does not gain priority over any student in J when priorities change from π to σ, any student in $(\mu^I)^{-1}(s^{\ell+1}) \cap J$ has higher priority than student $i^{\ell+1}$ under $\sigma_{s^{\ell+1}}$. Moreover for any $i^m \in \{i^1, \ldots, i^\ell\}$ with $i^m \in \nu_\ell^{-1}(s^{\ell+1})$, we must have $s^{\ell+1} = s^m$ and thus student i^m has higher priority than student $i^{\ell+1}$ at school $s^{\ell+1} = s^m$ under $\sigma_{s^{\ell+1}}$ by the choice of blocking pairs. Therefore student $i^{\ell+1}$ has lower priority under $\sigma_{s^{\ell+1}}$ than any student in $\nu_\ell^{-1}(s^{\ell+1})$ and hence pair $(i^{\ell+1}, s^{\ell+1})$ can block ν_ℓ under (P, σ) only if school $s^{\ell+1}$ has an empty seat at ν_ℓ.

Case 2: $i^{\ell+1} \in I \setminus J$ and $i^{\ell+1} \notin F_{s^{\ell+1}}(P)$.

By construction $\nu_\ell^{-1}(s^{\ell+1}) \subseteq J \cup \{i^1, \ldots, i^\ell\}$. Student $i^{\ell+1}$ has lower priority under $\sigma_{s^{\ell+1}}$ than any student in J. Moreover for any $i^m \in \{i^1, \ldots, i^\ell\}$ with $i^m \in \nu_\ell^{-1}(s^{\ell+1})$, we must have $s^{\ell+1} = s^m$ and thus student i^m has higher priority than student $i^{\ell+1}$ at school $s^{\ell+1} = s^m$ under $\sigma_{s^{\ell+1}}$ by the choice of blocking pairs. Therefore student $i^{\ell+1}$ has lower priority under $\sigma_{s^{\ell+1}}$ than any student in $\nu_\ell^{-1}(s^{\ell+1})$ and hence pair $(i^{\ell+1}, s^{\ell+1})$ can block ν_ℓ under (P, σ) only if school $s^{\ell+1}$ has an empty seat at ν_ℓ.

Case 3: $i^{\ell+1} \in I \setminus J$ and school $s^{\ell+1}$ and $i^{\ell+1} \in F_{s^{\ell+1}}(P)$.

Recall that $\nu_\ell^{-1}(s^{\ell+1}) \subseteq [(\mu^I)^{-1}(s^{\ell+1}) \cap J] \cup \{i^1, \ldots, i^\ell\}$. First suppose $\mu^I(i^{\ell+1}) \neq s^{\ell+1}$. For any $i^m \in \{i^1, \ldots, i^\ell\}$ with $i^m \in \nu_\ell^{-1}(s^{\ell+1})$, we must have $s^{\ell+1} = s^m$ and thus student i^m has higher priority than student $i^{\ell+1}$ at school $s^{\ell+1} = s^m$ under $\sigma_{s^{\ell+1}}$ by the choice of blocking pairs. Moreover $s^{\ell+1}$ is $i^{\ell+1}$'s first choice and yet μ^I is stable under (P, π). Therefore all students in $(\mu^I)^{-1}(s^{\ell+1})$ has higher priority under $\pi_{s^{\ell+1}}$ than $i^{\ell+1}$ does. But student $i^{\ell+1}$ does not gain priority at school $s^{\ell+1}$ over any student in J when priorities change from π to σ. Therefore any student in $(\mu^I)^{-1}(s^{\ell+1}) \cap J$ has higher priority under $\sigma_{s^{\ell+1}}$ than student $i^{\ell+1}$ does, which in turn implies any student in $\nu_\ell^{-1}(s^{\ell+1})$ has higher priority under $\sigma_{s^{\ell+1}}$ than student $i^{\ell+1}$ does. Hence pair $(i^{\ell+1}, s^{\ell+1})$ can block ν_ℓ under $\sigma_{s^{\ell+1}}$ only if school $s^{\ell+1}$ has an empty seat at ν_ℓ.

Next suppose $\mu^I(i^{\ell+1}) = s^{\ell+1}$. Let $i^m \in \{i^1, \ldots, i^\ell\}$ be such that $i^m \in \nu_\ell^{-1}(s^{\ell+1})$. We have $s^{\ell+1} = s^m$ and since student i^m has higher priority than student $i^{\ell+1}$ at school $s^{\ell+1}$ under $\sigma_{s^{\ell+1}}$, the same should be true under $\pi_{s^{\ell+1}}$ as well. That is because, student $i^{\ell+1}$ does not lose priority to any student at school $s^{\ell+1}$ when priorities change from $\pi_{s^{\ell+1}}$ to $\sigma_{s^{\ell+1}}$.

Since pair $(i^m, s^m) = (i^m, s^{\ell+1})$ blocks matching ν_{m-1} under (P, σ), and since assignments only improve as we proceed by the sequence $\nu_0, \nu_1, \ldots, \nu_k$,

$$s^{\ell+1} \; P_{i^m} \; \nu_{m-1}(i^m) \; R_{i^m} \; \nu_0(i^m).$$

So on one hand $\mu^I(i^{\ell+1}) = s^{\ell+1}$ where $i^{\ell+1}$ has lower priority at $s^{\ell+1}$ than i^m under $\pi_{s^{\ell+1}}$, and on the other hand i^m strictly prefers $s^{\ell+1}$ to its assignment under ν_0. That means

(1) $i^m \in I \setminus J$,

(2) $I^m \in F_{s^{\ell+1}}(P)$, and

(3) $\mu^I(i^m) = s^{\ell+1}$.

(1) holds because $i^m \in J$ would imply $\nu_0(i^m) = \mu^I(i^m)$ and in that case pair $(i^m, s^{\ell+1})$ would have blocked matching μ^I under (P, π). (2) holds because i^m has higher priority than $i^{\ell+1}$ at school $s^{\ell+1}$ under $\sigma_{s^{\ell+1}}$ although school $s^{\ell+1}$ is the first choice of student $i^{\ell+1}$. (3) holds because if $\mu^I(i^m) \neq s^{\ell+1}$, then by (2) $s^{\ell+1} \, P_{i^m} \, \mu^I(i^m)$ and in that case pair $(i^m, s^{\ell+1})$ would have blocked matching μ^I under (P, π).

So for each $i^m \in \nu_\ell^{-1}(s^{\ell+1})$, there is one empty seat at $s^{\ell+1}$ under ν_0. In addition there is at least one more empty seat, namely the seat student $i^{\ell+1}$ occupies at school $s^{\ell+1}$ at matching μ^I. Therefore under matching ν_ℓ there must still be at least one empty seat at school $s^{\ell+1}$.

This covers all three cases and completes the proof of Claim 2. $\diamondsuit$

We are now ready to complete the proof. Since each student weakly prefers and one strictly prefers matching ν_ℓ to matching $\nu_{\ell-1}$ for any $\ell \geq 0$, eventually the sequence terminates which means no pair blocks the final matching ν_k in the sequence and thus ν_k is stable under (P, σ). Therefore for any student $j \in J$,

$$\nu^I(j) \; R_j \; \nu_k(j) \; R_j \; \underbrace{\nu_0(j)}_{=\mu^I(j)}$$

where the first relation holds by the definition of the student-optimal stable matching. This completes the proof. $\square$

Proof of Proposition 3: Let μ^I, ν^I be the student-optimal stable matching for economies (P, π), $(P, \tilde{\pi})$ respectively. We have to show that $\nu^I(j) \, R_j \, \mu^I(j)$ for any $j \in M$. For any school s, the priority order $\tilde{\pi}_s$ is such that:

1. any student in $M \cup F_s(P)$ has higher priority under $\tilde{\pi}_s$ than any student in $I \setminus (M \cup F_s(P))$, and

2. for any student $j \in M \cup F_s(P)$ and any student $i \in I$, if j has higher priority than i under π_s then j also has higher priority than i under $\tilde{\pi}_s$.

Therefore $\nu^I(j)\, R_j\, \mu^I(j)$ for any $j \in M$ by Lemma 1. □

Proof of Proposition 4: Fix an economy (P, π). Let $M_1 \subset I$ be the set of sophisticated students and N_1 be the set of sincere students. Next consider an initially sincere student $i \in N_1$ and suppose she becomes sophisticated. Let $M_2 = M_1 \cup \{i\}$ be the set of sophisticated students including i, and let $N_2 = N_1 \setminus \{i\}$ be the set of remaining sincere students. Let ν^I be the Pareto-dominant Nash equilibrium of the Boston game where M_1 and N_1 are the sophisticated and sincere players, respectively. Let μ^I be the Pareto-dominant Nash equilibrium of the Boston game where M_2 and N_2 are the sophisticated and sincere players respectively. Let $\tilde{\mu}^1$ be the augmented priority ordering when M_1 is the set of sophisticated students and $\tilde{\mu}^2$ be the augmented priority ordering when M_2 is the set of sophisticated students. By Proposition 1, ν^I is the student-optimal stable matching for economy $(P, \tilde{\mu}^1)$ and μ^I is the student-optimal stable matching for economy $(P, \tilde{\mu}^2)$.

By the construction of the augmented priorities, student i does not lose priority to any student when priorities change from $\tilde{\mu}^1$ to $\tilde{\mu}^2$ while priorities between other students remain the same between $\tilde{\mu}^1$ and $\tilde{\mu}^2$. Therefore $\mu^I(i)\, R_i\, \nu^I(i)$ immediately follows from Balinski and Sönmez (1999). Moreover by construction of the augmented priorities, for any school s:

1. any student in $M_1 \cup F_s(P)$ has higher priority under $\tilde{\pi}^1_s$ than any student in $I \setminus (M_1 \cup F_s(P))$, and

2. for any student $j \in M_1 \cup F_s(P)$ and any other student $h \in I$, if j has higher priority than h under $\tilde{\pi}^2_s$ then j also has higher priority than h under $\tilde{\pi}^1_s$.

Therefore $\nu^I(j)\, R_j\, \mu^I(j)$ for any $j \in M_1$ by Lemma 1. □

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